I. (20) State and prove the Monotone Convergence Theorem for sequences. (You may give the proof in the case that the sequence is monotone increasing.)

**Solution:** The Theorem says that every bounded monotone sequence in \( \mathbb{R} \) converges to some real number \( a \). If \( \{x_n\} \) is monotone increasing, let \( a = \text{lub}\{x_n \mid n \geq 1\} \). Then for all \( \varepsilon > 0 \), \( a - \varepsilon \) is not an upper bound for \( \{x_n \mid n \geq 1\} \). Hence there exists some \( n_0 \) such that \( a - \varepsilon < x_{n_0} \leq a \). Since the sequence is monotone increasing, for all \( n \geq n_0 \), we have \( a - \varepsilon < x_n \leq a \). But this implies that \( |x_n - a| < \varepsilon \) for all \( n \geq n_0 \). Hence \( x_n \to a \).

B) (10) Suppose that \( x_n \) is a sequence of strictly negative numbers and \( x_{n+1}/x_n \leq 1 \) for all \( n \geq 1 \). Show that \( \lim_{n \to \infty} x_n = a \) exists in \( \mathbb{R} \) and satisfies \( a \leq 0 \).

**Solution:** Since the \( x_n < 0 \), from the inequality \( x_{n+1}/x_n \leq 1 \), we get \( x_{n+1} \geq x_n \) for all \( n \geq 1 \). (The inequality reverses when we multiply by the negative number \( x_n \).) This implies the sequence is monotone increasing. Since it is also bounded above by 0, part (A) shows that \( x_n \to a \) for some \( a \leq 0 \).

II. (20) Suppose \( \{x_n\} \) is a sequence such that \( |x_n - 10| < 100 \) for all \( n \geq 1 \). Show that there exists some number \( a \in [-90, 110] \) and a subsequence \( \{x_{n_k}\} \) such that \( x_{n_k} \to a \). State any “big theorems” you are using.

**Solution:** Since \( |x_n - 10| < 100 \) for all \( n \), it follows that \(-100 < x_n - 10 < 100\), so \(-90 < x_n < 110\). Hence \( |x_n| \leq 110 \) for all \( n \), so this is a bounded sequence. By the Bolzano-Weierstrass theorem, there must be a convergent subsequence \( x_{n_k} \to a \) for some \( a \in \mathbb{R} \). The limit must also lie in the interval \([-90, 110]\) because of the “order limit theorem” (Corollary 2.2.8 in our text). This result says: If \( a \leq x_n \leq b \) for all \( n \), and \( x_n \to c \), then \( a \leq c \leq b \) also.

III. (20) Show using the \( \varepsilon, \delta \) definition that \( \lim_{x \to 2} x^2 - x = 2 \).

**Solution:** (“prep”: \( |x^2 - x - 2| = |x - 2||x + 1| \). If \( |x - 2| < 1 \), then \( 1 < x < 3 \), so \( |x + 1| \) satisfies \( 2 < |x + 1| < 4 \).)

**proof:** Given \( \varepsilon > 0 \), let \( \delta = \min\left(1, \frac{\varepsilon}{4}\right) \). Then for all \( x \) with \( 0 < |x - 2| < \delta \), \( 1 < x < 3 \), so \( |x + 1| < 4 \). Therefore, for all such \( x \),

\[
|x^2 - x - 2| = |x - 2||x + 1| < \frac{\varepsilon}{4} \cdot 4 = \varepsilon.
\]

This shows \( \lim_{x \to 2} x^2 - x = 2 \).
IV. True/False. For each true statement, give a short proof. For each false statement, give a counterexample or reason.

A) (10) If \( \{x_n\} \) diverges, then for all \( a \in \mathbb{R} \), there exist \( \varepsilon_0 > 0 \) and \( n_0 \in \mathbb{N} \) such that \( |x_n - a| \geq \varepsilon_0 \) for all \( n \geq n_0 \).

Solution 1: This is FALSE. A counterexample is the sequence \( x_n = (-1)^n \). If \( a = 1 \) (or \( a = -1 \)), note there is no \( n_0 \) and no \( \varepsilon_0 > 0 \) such that \( |x_n - a| \geq \varepsilon_0 \) for all \( n \geq n_0 \) because \( |(-1)^n - 1| = 0 \) for all even \( n \). (Similarly, \( |(-1)^n + 1| = 0 \) for all odd \( n \).)

Solution 2: We can also see that this is FALSE if we think of the meaning of the statement: For all \( a \in \mathbb{R} \), \( x_n \not\to a \) (that is, \( x_n \) does not converge to \( a \)). In symbolic form, by negating the definition of the statement \( x_n \to a \), this says

\[
\forall a, \exists \varepsilon_0 > 0, \forall n_0, \exists n \geq n_0, \text{ s.t. } |x_n - a| \geq \varepsilon_0.
\]

(Note: the quantifier on the \( n \) is \( \exists \), not \( \forall \). The given statement would certainly imply that the sequence does not converge to \( a \), but it is not necessarily true for divergent sequences, as the example in the first solution shows.)

B) (10) It is possible to find a function \( f : \mathbb{R} \to \mathbb{R} \) that is continuous at \( c = \sqrt{2} \), with \( f(\sqrt{2}) = 0 \), but satisfying \( f(x) = 1 \) for all rational \( x \).

Solution 1: This is FALSE. If \( \{x_n\} \) is any sequence of rational numbers with \( x_n \to \sqrt{2} \), then by a standard property of continuous functions, \( f(x_n) \to f(\sqrt{2}) \). But \( f(x_n) = 1 \) for all \( n \), and \( f(\sqrt{2}) = 0 \). So this is not possible.

Solution 2: We can also see that this is FALSE if we apply the definition of continuity directly. Aiming for a contradiction, suppose there were such a function. Then for all \( \varepsilon > 0 \), there would exist \( \delta > 0 \) such that \( |f(x) - f(\sqrt{2})| < \varepsilon \) for all \( x \) with \( |x - \sqrt{2}| < \delta \). If \( f(\sqrt{2}) = 0 \) and we take \( \varepsilon < 1 \), then the interval defined by \( |y - 0| < \varepsilon \) does not contain 1. But every interval defined by \( |x - \sqrt{2}| < \delta \) for \( \delta > 0 \) contains rational numbers \( x \) and for those \( x \), \( f(x) = 1 \). This is a contradiction which shows no such \( f \) exist.

C) (10) The sequence defined by \( x_1 = 1 \) and \( x_n = \sqrt{3x_{n-1} + 1} \) for \( n \geq 2 \) is monotone increasing.

Solution: This is TRUE. We have \( x_1 = 1 \) and \( x_2 = \sqrt{4} = 2 \), so \( x_2 > x_1 \). This is the base case for an induction. Assume \( x_{k+1} \geq x_k \) for a given \( k \). Then by the induction hypothesis,

\[
x_{k+2} = \sqrt{3x_{k+1} + 1} \geq \sqrt{3x_k + 1} = x_{k+1}.
\]

Therefore \( \{x_n\} \) is increasing by induction.

Extra Credit. (10) Assume that \( \{x_n\} \) is a sequence that converges to \( a \). Construct a new sequence \( \{y_n\} \) by making \( y_n \) the average of the first \( n \) terms in \( \{x_n\} \): \( y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \). Show that \( y_n \) also converges to \( a \).
Solution: Here’s the idea. By the triangle inequality, note that

$$|y_n - a| = \left| \frac{x_1 + x_2 + \cdots + x_n}{n} - a \right|$$

$$= \frac{1}{n} |x_1 + x_2 + \cdots + x_n - na|$$

$$\leq \frac{1}{n} (|x_1 - a| + \cdots + |x_n - a|).$$

Since \( \{x_n\} \) converges to \( a \), recall that we know the sequence is bounded. This implies that there exists \( M \) such that \( |x_n - a| \leq M \) for all \( n \) as well. Now, given \( \varepsilon > 0 \), there exists \( N \) such that \( |x_n - a| < \varepsilon/2 \) for all \( n \geq N \). So if \( n \geq N \), continuing from the last line above,

$$\leq \frac{1}{n} (|x_1 - a| + \cdots + |x_{N-1} - a|) + \frac{n - N + 1}{n} \frac{\varepsilon}{2}$$

$$< \frac{NM}{n} + \frac{\varepsilon}{2}.$$

Since we can think of the \( N \) that works here as fixed, but \( n \) is still allowed to grow, note that we can now find \( n \) large enough so that \( \frac{NM}{n} < \frac{\varepsilon}{2} \) as well by taking \( n > \frac{2NM}{\varepsilon} \).

In other words, given \( \varepsilon > 0 \), we take \( n_0 = \max(N, \frac{2NM}{\varepsilon}) \). Then \( n \geq n_0 \) implies \( |y_n - a| < \varepsilon \).