I.

A) State the Axiom of Completeness for the real number system.

_Solution:_ Every nonempty set of real numbers that is bounded above has a least upper bound. (This is an _axiom_ – we take it as an unproved assertion describing properties of the real number system.)

B) Prove that if $A$ and $B$ are bounded sets of real numbers with $\text{lub}(A) < \text{lub}(B)$, then there exists a single $y \in B$ satisfying $y > x$ for all $x \in A$.

_Solution:_ Since $\text{lub}(A) < \text{lub}(B)$, $\text{lub}(A)$ is not an upper bound for $B$. Hence there exists at least one $y_0 \in B$ such that $\text{lub}(A) < y_0 \leq \text{lub}(B)$. (The subscript is one way mathematicians often indicate that they are referring to one particular element of a set.) But then, by definition, every $x \in A$ satisfies $x \leq \text{lub}(A)$. So $x < y_0$ for all $x \in A$.

C) Let $A = \cap_{n=1}^{\infty} \left( -\frac{1}{2n}, 1 + \frac{1}{2n} \right)$. Explain why $A$ is bounded below and determine $\text{glb}(A)$.

_Solution:_ It is certainly true that $-\frac{1}{2}$ is a lower bound for $A$. But in fact, given any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ such that $\frac{1}{2n} < \varepsilon$, and hence $-\frac{1}{2n} > -\varepsilon$. We can see that $A = [0, 1]$, so $\text{glb}(A) = 0$.

II.

A) Prove that every interval $(a, b)$ with $0 < a < b$ contains a rational number $\frac{m}{n}$.

_Solution:_ Since $\mathbb{N}$ is not bounded, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$, or equivalently $\frac{1}{n} < b - a$. Pick any one such $n$ and fix it. For that fixed $n$, let $M = \{m \in \mathbb{N} \mid m > na\}$. This set is not empty (again since $\mathbb{N}$ is not bounded). So by the Well-Ordering Property, it contains a smallest element. Call that $m_0$. Now $m_0 > na$, so $\frac{m_0}{n} > b$. On the other hand, $m_0 - 1 \notin M$, so $m_0 - 1 \leq na$. This implies $\frac{m_0}{n} \leq a + \frac{1}{n} < a + b - a = b$. Combining these last facts shows that the rational number $\frac{m_0}{n} \in (a, b)$.

B) Use part A to show that every interval $(a, b)$ with $a < b < 0$ contains a rational number $\frac{m}{n}$.

_Solution:_ If $a < b < 0$, then $0 < (-b) < (-a)$. By part A, there is a rational number $\frac{-m}{n}$ satisfying $(-b) < \frac{-m}{n} < (-a)$. But then, multiplying these inequalities by $-1$, $a < \frac{m}{n} < b$. Since $\frac{m}{n}$ is also rational, we are done.

III. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(1) = 2$ and $f(n + 1) = \frac{2f(n) - 1}{3}$ for all $n \geq 1$. Using mathematical induction, show that $f(n) > -1$ for all $n \geq 1$. 

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Solution: The base case is \( n = 1 \). By definition, \( f(1) = 2 > -1 \). Now assume as
the induction hypothesis that \( f(k) > -1 \), and consider \( f(k + 1) = \frac{2f(k) - 1}{3} \). Then the
properties of order imply

\[
2f(k) > -2
\]

hence \( 2f(k) - 1 > -3 \)

hence \( f(k + 1) = \frac{2f(k) - 1}{3} > -1, \)

which is what we wanted to show. By the principle of mathematical induction, \( f(n) > -1 \) for all \( n \geq 1 \).

IV. True-False. For each true statement give a short proof or reason; for each false statement give a counterexample.

A) If \( A \) and \( B \) are sets of real numbers with the property that \( a > b \) for all \( a \in A \) and all
\( b \in B \), then \( \text{glb}(A) > \text{lub}(B) \).

Solution: This is FALSE. A counterexample – let \( B = (-\infty, 0) \) and \( A = (0, \infty) \) in \( \mathbb{R} \).
Then \( a > b \) for all \( a \in A \) and all \( b \in B \). But \( \text{glb}(A) = 0 = \text{lub}(B) \).

B) \( (10) \) For all \( n \geq 1 \), \( \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k} = \frac{3^n}{2^n} \).

Solution: This is TRUE. By the binomial theorem,

\[
\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x + y)^n
\]

for all real numbers \( x, y \). We get the desired equality by making \( x = \frac{1}{2}, y = 1 \).

C) \( (10) \) The smallest \( c \) such that \( \{ x \in \mathbb{R} \mid |x + 1| + |x + 10| = c \} \) is not empty is \( c = 11 \).

Solution: This is FALSE. \( |x + 1| + |x + 10| \) represents the sum of the distance from
\( x \) to \(-1\) and the distance from \( x \) to \(-10\) along the number line. For any \( x \) with
\(-10 \leq x \leq -1\), the sum of these distances will be \( 9 < 11 \).

Extra Credit (10) Caution: this problem may be “habit forming.” Only attempt after
finishing the rest of the exam! Can you find a 1-1, onto function \( f : \mathbb{R} \rightarrow (-1, 1) \)? If so,
describe one by giving a formula or a graph. If there is no such function, prove it.

Solution: There is such a function. We can construct one as follows: First take any strictly
increasing onto function \( f : \mathbb{R} \rightarrow (-1, 1) \) such as \( f(x) = \frac{2}{\pi} \arctan(x) \). This “misses” 1, of
course, so we have to modify \( f \) to get 1 in the range. If we define a new function \( g \) with
\( g(1) = 1 \), but \( g(x) = f(x) \) for all \( x \neq 1 \), then we have missed \( f(1) = \frac{1}{2} \) in the range of \( f \).
But we can make $g(2) = f(1) = \frac{1}{2}$, then $g(3) = f(2)$, etc. to get a function that is still 1-1 and onto:

$$
\begin{align*}
  g(x) &= f(x) = \frac{2}{\pi} \arctan(x) \quad \text{if } x \notin \mathbb{N} \\
  g(1) &= 1 \\
  g(n) &= f(n - 1) \quad \text{if } n \geq 2 \in \mathbb{N}
\end{align*}
$$

(This is a mathematical version of the “Hilbert Hotel” with rooms numbered by $n \in \mathbb{N}$ – there’s always room for one more guest if we shift everyone else down one room!)