Mathematics 242 – Principles of Analysis
Exam 1 – February 25, 2010

Directions: Read all the questions *carefully* before starting to work. Do all work in the blue exam booklet. There are 100 possible regular points and 10 possible Extra Credit points.

I.

A) (10) State the LUB Axiom for the real number system.

B) (10) Prove that if A and B are bounded sets of real numbers with $\text{lub}(A) < \text{lub}(B)$, then there exists a single $y \in B$ satisfying $y > x$ for all $x \in A$.

C) (10) Let $A = \bigcap_{n=1}^{\infty} \left(\frac{-1}{2n}, 1 + \frac{1}{2n}\right)$. Explain why A is bounded below and determine $\text{glb}(A)$.

II.

A) (20) Prove that every interval (a, b) with $0 < a < b$ contains a rational number $\frac{m}{n}$.

B) (5) Use part A to show that every interval (a, b) with $a < b < 0$ contains a rational number $\frac{m}{n}$.

III. (15) Let $f : \mathbf{N} \to \mathbf{R}$ be defined by $f(1) = 2$ and $f(n+1) = \frac{2f(n)-1}{3}$ for all $n \geq 1$. Using mathematical induction, show that $f(n) > -1$ for all $n \geq 1$.

IV. True-False. For each true statement give a short proof or reason; for each false statement give a counterexample.

A) (10) If A and B are sets of real numbers with the property that $a > b$ for all $a \in A$ and all $b \in B$, then $\text{glb}(A) > \text{lub}(B)$.

B) (10) For all $n \geq 1$, $\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k} = \frac{3^n}{2^n}$.

C) (10) The smallest c such that $\{x \in \mathbf{R} \mid |x+1| + |x+10| = c\}$ is not empty is $c = 11$.

Extra Credit (10)

Caution: this problem may be "habit forming." Only attempt after finishing the rest of the exam!

Can you find a 1-1, onto function $f : \mathbf{R} \to (-1, 1]$? If so, describe one by giving formulas or a graph. If there is no such function, prove it.