Mathematics 242 – Principles of Analysis
Information on Final Examination
May 9, 2011

General Information

- The final examination for this class will be given during the scheduled period – 3:00 to 5:30 pm on Tuesday, May 17.
- The final will be a *comprehensive exam*, covering all the topics from the three midterms, *and the material about series from the past week.* See the list of topics below for more details.
- The exam will be similar in format to the midterms but 1.5 to 1.75 times as long – it will be written to take about 1.5 hours = 90 minutes if you work steadily, but you will have the full 2.5 hour = 150 minute period to use if you need that much time.
- If there is interest, I would be happy to arrange an evening review session during exam week – I think I'm free every evening. We can discuss this in class on May 9.

Philosophical Comments and Suggestions on How to Prepare

- The reason we give final exams in almost all mathematics classes is to encourage students to "put whole courses together" in their minds. Also, preparing for the final should help to make the ideas "stick" so you will have the material at your disposal to use in later courses.
- If you approach preparing for a final exam in the right way it can be a *real learning experience* – especially in a class like this one where almost everything we have done "fits together" in a very tight chain of logical reasoning starting with the Completeness Axiom for the real number system. Much of what we did earlier in the semester may and should make much more sense now than it may have the first time around!
- *Start reviewing now*, and do some review each day between now and May 17 (even just 1/2 hour each day will make a big difference). That way you will not be "crunched" at the end (and with any luck the ideas we have developed in this course will "stick" better!)

Topics To Be Included

0) Logic, sets, functions
1) The real number system, rational and irrational numbers, the algebraic and order properties, least upper bounds (Axiom of Completeness)
2) Mathematical induction
3) Sequences, convergence ($\lim_{n\to\infty} x_n$ – both the ε, n_0 definition, and computing limits via the limit theorems).
4) Subsequences, The Nested Interval Theorem, and the Bolzano-Weierstrass theorem
5) Limits of functions (the ε, δ definition), the algebraic and order limit theorems, "squeeze theorem,"
6) Continuity, the Extreme and Intermediate Value Theorems

7) Definition and properties of the derivative, the Mean Value Theorem and its consequences
8) The definite integral, integrability, the Fundamental Theorem
9) Infinite series – convergence and divergence, key examples such as geometric series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$-series, etc. Absolute vs. conditional convergence. Comparison, alternating series, and ratio tests for convergence.

Proofs to Know

You should be able to give precise statements of *all the definitions* listed on the course homepage and the theorems mentioned in the outline above. Also, be able to give proofs of the following:

1) Every monotone increasing sequence of real numbers that is bounded above converges.
2) The Bolzano-Weierstrass Theorem
3) The Intermediate Value Theorem (the proof of the special case we did in class)
4) The Mean Value Theorem (including the special case known as "Rolle's Theorem;" the general statement is deduced from that).

Suggested Review Problems

See review sheets for Midterm Exams 1, 2, and 3 for topics 1-8 in the list above. (Those review sheets are now reposted on the course homepage if you need another copy.)

Practice Questions

I. Let $A = \{\cos(x) : x \in [0, 3\pi/4]\}$ and let $B = \{x : 1 < x^2 < 4\}$.

A) What is the set $A \cup B$?
B) What is the least upper bound of the set $C = \{|x - 2| : x \in A\}$?

II.

A) State the ε, n_0 definition for convergence of a sequence.
B) Identify $L = \lim_{n\to\infty} x_n$ for the sequence

$$x_n = \frac{3n^2 + n}{n^2 + 1}$$

and prove using the definition that $\lim_{n\to\infty} x_n = L$.

III. Let $x_n = \sin(2\pi \cos(n))$. Show that there exists a convergent subsequence of (x_n). (Don't try to find one explicitly!)

IV.

A) Is $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$ absolutely convergent, conditionally convergent, or divergent?

B) Same question as in A for $\sum_{k=0}^{\infty} \frac{(-1)^k k^3 3^k}{k!}$.

C) For which $x \in \mathbf{R}$ does the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

converge?

V.

A) Give the ε, δ definition for the statement $\lim_{x \to c} f(x) = L$.

B) Identify the limit

$$L = \lim_{x \to 0^+} x^{1/2} \sin(1/x)$$

and prove using the definition that $\lim_{x \to 0^+} x^{1/2} \sin(1/x) = L$.

VI. All parts of this problem refer to the function

$$f(x) = \frac{32x}{x^4 + 48}$$

A) What are $f(0)$ and $f(2)$ for this function?

B) Show using the Intermediate Value Theorem that for each k with $0 < k < 1$, the equation $f(x) = k$ has *at least two* solutions $x \in \mathbf{R}$, with $x > 0$.

C) Show, using the Mean Value Theorem, that if f is differentiable on an interval $I = (a, b)$ and $f'(x) \neq 0$ for all $x \in I$, then for each k the equation $f(x) = k$ has *at most one* solution with $x \in I$. (Hint: Prove the contrapositive.)

D) Show using part C that there are *exactly two* solutions of the equation $f(x) = k$ from part B for each k with $0 < k < 1$.

VII. In this question you may use without proof the summation formulas:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Show that $f(x) = x^2 + x - 1$ is integrable on $[a, b] = [0, 3]$ by considering upper and lower sums for f and determine the value of $\int_0^3 x^2 + x - 1\, dx$.

VIII. True - False. For each true statement, give a short proof or reason. For each false statement give a reason or a counterexample.

A) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with positive terms. If the partial sums s_N are bounded above by some B for all N, then $\sum_{n=1}^{\infty} a_n$ converges.

B) If f is differentiable on $[a, b]$ with $f'(a) > 0$, then there is an interval containing a on which f is increasing.

C) The function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational} \end{cases}$$

is *continuous* at $x = 0$.

IX. Let

$$f(x) = \begin{cases} \cos(2x) & \text{if } x < 0 \\ ax^2 + bx + c & \text{if } x \geq 0 \end{cases}$$

There is exactly one set of constants a, b, c for which $f'(0)$ and $f''(0)$ both exist. Find them.