Mathematics 242 – Principles of Analysis
Final Examination Solutions
May 17, 2011

I.

A) State the ε, n_0 definition of convergence for a sequence of real numbers.

Solution: The sequence x_n converges to $c \in \mathbf{R}$ if for every $\varepsilon > 0$, there exist $n_0 \in \mathbf{N}$ such that $|x_n - c| < \varepsilon$ for all $n \geq n_0$.

B) Identify $\lim_{n\to\infty} \frac{2n^2-2}{n^2+n}$.

Solution: We have

$$\begin{aligned} \lim_{n\to\infty} \frac{2n^2-2}{n^2+n} &= \lim_{n\to\infty} \frac{2n^2-2}{n^2+n} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{2-\frac{2}{n^2}}{1+\frac{1}{n}} \\ &= 2. \end{aligned}$$

C) Show that your result in part B is correct using the definition.

Solution: Let $\varepsilon > 0$ and take $n_0 > 2/\varepsilon$. Then for all $n \geq n_0$, $2/n < \varepsilon$, so

$$\begin{aligned} \left| \frac{2n^2-2}{n^2+n} - 2 \right| &= \left| \frac{-2n-2}{n^2+n} \right| \\ &= \frac{2n+2}{n^2+n} \\ &= \frac{2}{n} \\ &< \varepsilon. \end{aligned}$$

Therefore, by definition, $\frac{2n^2-2}{n^2+n}$ converges to 2.

II. Show using mathematical induction that the sequence $\{x_n\}$ defined by $x_1 = 1$, and

$$x_n = \sqrt{x_{n-1}+1}, \qquad \text{if } n \geq 2$$

is bounded above by 2 and monotone increasing. Does this imply that $\{x_n\}$ is convergent? If so, say why and find the limit. If not, say why not.

Solution: By assumption $x_1 = 1 < 2$. Then if $x_k < 2$, we have $x_{k+1} = \sqrt{x_k+1} < \sqrt{2+1} < \sqrt{3} < 2$. Hence $x_n < 2$ for all n by induction.

Similarly, $x_1 = 1$ and $x_2 = \sqrt{2}$ satisfy $x_1 < x_2$. Assume that $x_k < x_{k+1}$. Then by definition

$$x_{k+1} = \sqrt{x_k+1} < \sqrt{x_{k+1}+1} = x_{k+2}.$$

Hence the sequence is monotone strictly increasing by induction.

Since the sequence is monotone increasing and bounded above, the monotone convergence theorem implies that $\{x_n\}$ converges to some c in $\mathbf{R}$ (indeed, $c = \text{lub}\{x_n \mid n \geq 1\}$). From the recurrence relation, taking $\lim_{n\to\infty}$, and using the continuity of the square root function,

$$c = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{x_{n-1}+1} = \sqrt{c+1}.$$

This implies $c^2 = c+1$, so by the quadratic formula, $c = \frac{1\pm\sqrt{5}}{2}$. Since all the terms $x_n > 0$, this implies

$$c = \frac{1+\sqrt{5}}{2} \doteq 1.618.$$

III. The Mean Value Theorem states: If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof: First, consider the case $f(b) = f(a)$. If f is constant on $[a,b]$, then we can take $c \in (a,b)$ arbitrary. Otherwise, by the Extreme Value Theorem, f must attain a maximum or minimum not equal to $f(a) = f(b)$ at some $c \in (a,b)$. Hence f has a local maximum or minimum at c so it follows from the definition of the derivative that $f'(c) = 0$. This establishes the special case where $f(b) = f(a)$.

Now consider a general f satisfying the conditions of the theorem, and consider

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a).$$

The function g is a linear combination of $f(x)$ and the linear function $x-a$. Hence g is also continuous on $[a,b]$ and differentiable on (a,b). Moreover, $g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$ and $g(a) = f(a) - 0 = f(a)$. Therefore, g satisfies the extra condition of the special case. It follows that there exists a $c \in (a,b)$ where $g'(c) = 0$. But $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$, so $f'(c) = \frac{f(b)-f(a)}{b-a}$, and the theorem is proved.

IV. True/False. For each true statement, give a short proof. For each false statement, give a counterexample.

A) Let f be differentiable and f' be continuous on $[a,b]$. If $f'(x) \neq 0$ for all $x \in [a,b]$, then $f(x)$ is either increasing on the whole interval $[a,b]$, or decreasing on the whole interval $[a,b]$.

   Solution: This is TRUE. Since f' is continuous on $[a,b]$, the Extreme and Intermediate Value Theorems can be applied to f'. Since f' never takes the value 0 on $[a,b]$, either the minimum value of f' is strictly greater than zero, or else the maximum value of f' is strictly less than zero. In the first case, by one of our consequences of the Mean Value Theorem, f is increasing on $[a,b]$. In the second case, f is decreasing on $[a,b]$.

B) If f is uniformly continuous on $[a, b]$, then f is differentiable at every point in $[a, b]$.

Solution: This is FALSE. By a theorem we proved in class, any continuous function on $[a, b]$ is uniformly continuous. But there are continuous functions that are not differentiable at some $c \in [a, b]$. An example is $f(x) = |x|$ on $[a, b] = [-1, 1]$. f is continuous, but is not differentiable at $x = 0$.

C) Let $f : [a, b] \to \mathbf{R}$ be continuous and let $\{x_n\}$ be a sequence with $x_n \in [a, b]$ for all n. Then The sequence $\{f(x_n)\}$ has a convergent subsequence.

Solution: This is TRUE. By the Extreme Value Theorem, the sequence $\{f(x_n)\}$ is bounded (it lies in the interval $[m, M]$ where m, M are the minimum and maximum values of f on the interval). Then the Bolzano-Weierstrass theorem implies that it has a convergent subsequence.

V. Show that

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ 2 & \text{if } 1/2 \le x \le 1 \end{cases}$$

is integrable on $[a, b] = [0, 1]$ by considering the upper and lower sums for f. Also, determine the value of $\int_0^1 f$.

Solution: Let $\delta < 1/2$ and let $\mathcal{P}_\delta$ be the partition $\{0, 1/2 - \delta, 1/2 + \delta, 1\}$. Then

$$\begin{aligned} L_{\mathcal{P}_\delta}(f) &= 1 \cdot (1/2 - \delta) + 1 \cdot (2\delta) + 2 \cdot (1/2 - \delta), \\ U_{\mathcal{P}_\delta}(f) &= 1 \cdot (1/2 - \delta) + 2 \cdot (2\delta) + 2 \cdot (1/2 - \delta), \\ U_{\mathcal{P}_\delta}(f) - L_{\mathcal{P}_\delta}(f) &= 2\delta. \end{aligned}$$

Given $\varepsilon > 0$, let $\delta < \varepsilon/2$. Then for any such δ, the difference between the upper and lower sums is $< \varepsilon$. This implies that f is integrable on $[0, 1]$. This also shows

$$\int_0^1 f = \text{glb}\{U_{\mathcal{P}_\delta}(f) \mid \delta > 0\} = \frac{3}{2},$$

since

$$U_{\mathcal{P}_\delta}(f) = 1 \cdot (1/2 - \delta) + 2 \cdot (2\delta) + 2 \cdot (1/2 - \delta) = \frac{3}{2} + \delta.$$

VI.

A) Noting that

$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = \lim_{x \to 1} \frac{\ln(x) - \ln(1)}{x - 1},$$

evaluate the limit. (You may use any calculus facts you need here without proof.)

Solution: Let $f(x) = \ln(x)$. By definition, the limit above computes the derivative $f'(1)$. Since $f'(x) = 1/x$ by calculus facts, $f'(1) = 1$, so

$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1.$$

B) Let $x = 1 + \frac{1}{n}$ in the limit from part A and use that result and the sequential criterion for continuity applied to $f(x) = \exp(x) = e^x$ to compute

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

Solution: With $x = 1 + \frac{1}{n} \to 1$, we have, using properties of logarithms and continuity of the exponential function,

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \ln\left(\left(1 + \frac{1}{n}\right)^n\right) \\ \Rightarrow e^1 &= \lim_{n\to\infty} \exp\left(\ln\left(1 + \frac{1}{n}\right)^n\right). \end{aligned}$$

Hence

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

VII.

A) State the definition of convergence for an infinite series $\sum_{n=1}^{\infty} a_n$.

Solution: The series $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{s_k\}$ of partial sums, defined by

$$s_k = \sum_{n=1}^{k} a_n,$$

converges to some $S \in \mathbf{R}$ (as a sequence).

B) Does the series $\sum_{n=1}^{\infty} \frac{2^n}{5^n}$ converge? If so, what is the sum of the series?

Solution: The answer is yes – this is a geometric series with ratio $r = 2/5$. Since $|r| < 1$, the geometric series converges to

$$\frac{2/5}{1 - 2/5} = \frac{2}{3}.$$

C) Does the series $\sum_{n=1}^{\infty} ne^{-n}$ converge? (Use the Integral Test.)

Solution: The terms of the series are $f(n)$ for $f(x) = xe^{-x}$. (Note that $f'(x) = e^{-x}(x-1) < 0$ for all $x > 1$. So $f(x)$ is decreasing to 0 as $x \to \infty$.) Integrating by parts and using the Fundamental Theorem of Calculus, we have

$$\begin{aligned} \lim_{b\to\infty} \int_1^b xe^{-x}\, dx &= \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_1^b \\ &= \lim_{b\to\infty} -be^{-b} + e^{-b} - e^{-b} + 2e^{-1} \\ &= 2e^{-1} \end{aligned}$$

Since this is finite, the series *converges.*

D) Use the Ratio Test to determine the $x \in \mathbf{R}$ for which the series $\sum_{n=1} \frac{nx^n}{2^n}$ converges absolutely.

Solution: We have

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{nx^n} \right| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \frac{|x|}{2} = \frac{|x|}{2}.$$

This is < 1 if and only if $|x| < 2$, or $x \in (-2, 2)$. The series converges absolutely for $x \in (-2, 2)$, but diverges for $x = \pm 2$.

Extra Credit Let

$$f = \begin{cases} x + x^2 \cos(\pi/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable at all $x \in \mathbf{R}$, but f' is not continuous at $x = 0$.

Solution: For $x \neq 0$, we can apply the usual derivative rules to see

$$f'(x) = 1 + 2x \cos(\pi/x) + \pi \sin(\pi/x).$$

For $x = 0$, we must use the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x + x^2 \cos(\pi/x)}{x} \\ &= \lim_{x\to 0} 1 + x \cos(\pi/x) \\ &= 1 \end{aligned}$$

(note that $\lim_{x\to 0} x \cos(\pi/x) = 0$ by the squeeze theorem). Hence $f'(x)$ exists for all $x \in \mathbf{R}$.

Finally, we claim that f' fails to be continuous at $x = 0$. this is true because

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 1 + 2x \cos(\pi/x) + \pi \sin(\pi/x)$$

does not exist. The last term, $\sin(\pi/x)$ has no limit as $x \to 0$, but the other two terms do. It follows that f' has a discontinuity at $x = 0$.