Mathematics 242 – Principles of Analysis
Review Sheet – Midterm Exam 3
April 29, 2011

General Information

As you know from the course syllabus, the third and final midterm exam for the course will be given in class on Friday, May 6. This be an individual closed book exam similar in format to the other midterms we have done this semester. I will be happy to hold a late afternoon or evening review session to help you prepare. Late afternoon times are possible Wednesday, May 4, but I will have to leave campus no later than 5:00pm to get to an off-campus commitment, so Wednesday evening will not work for me that week. Tuesday or Thursday evening after the department colloquium and majors' Cinco de Mayo dinner would be OK.

Topics to be Covered

The exam will cover the material we have covered since the last exam (problem sets 7,8,9). This is sections 3.5, 3.6, 4.1, 4.2, 4.3, 5.1, 5.2, 5.3, and 5.4 in the text, but as before, not all the topics in those sections were discussed in class. You are responsible for only what we did talk about:

1) Properties of continuous functions on closed intervals: The Intermediate Value and Extreme Value Theorems (IVT, EVT), uniform continuity. (Note: This overlaps Exam 2 to some extent, but it is also an important component of several other topics from this part of the course).
2) The definition of differentiability and examples.
3) Rolle's Theorem, the Mean Value Theorem (MVT) and its consequences.
4) The Cantor set and function.
5) The definition of integrability, criteria for integrability, computations of definite integrals from the definition.
6) The Fundamental Theorem of Calculus (FTC).

What to Expect

The exam will have four or five questions, each possibly with several parts. Some questions will ask for a precise statement of a definition or a theorem we have discussed. Be prepared to give careful statements of the definitions noted above and know how to use them (for instance how to show that a limit exists using the ε, δ definition). Also know and be able to give these proofs:

1) The Mean Value Theorem (including the special case known as Rolle's Theorem – deduce the general result from that).
2) A monotone increasing function on an interval $[a, b]$ is integrable.
3) Part 1 of the FTC: If f is continuous on $[a, b]$, and $F(x) = \int_a^x f$, then $F'(x) = f(x)$.

The other questions will be similar to questions from the problem sets and discussions.

Review Problems

A) For each of the following, give an example or a short proof that no such examples exist:

1) A function $g : \mathbf{R} \to \mathbf{R}$ that is continuous at no $x \in \mathbf{R}$.

2) A function $f : [0, 1] \to \mathbf{R}$ such that $3 = \text{lub}\{f(x) : x \in [0, 1]\}$, but there is no $x \in [0, 1]$ with $f(x) = 3$.

3) A continuous function $f : [0, 1] \to \mathbf{R}$ such that $3 = \text{lub}\{f(x) : x \in [0, 1]\}$, but there is no $x \in [0, 1]$ with $f(x) = 3$.

4) A continuous function $f : [0, 1] \to [0, 1]$ with $f(0) = 0$, $f(1) = 1$, but such that there is no $x \in [0, 1]$ with $f(x) = 1/3$.

5) A function $f : [a, b] \to \mathbf{R}$ that is differentiable on $[a, b]$, satisfies $f(a) = f((a+b)/2) = f(b)$, but has only one critical point in (a, b). (Recall, a critical point is a c where $f'(c) = 0$ or $f'(c)$ does not exist.)

6) A set of real numbers A such that A is uncountably infinite, but such that for all $\varepsilon > 0$, there exist intervals $I_n = [a_n, b_n]$ for $n \in \mathbf{N}$ such that $A \subset \cup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty}(b_n - a_n) < \varepsilon$.

7) A differentiable function $f : \mathbf{R} \to \mathbf{R}$ such that $f'(x) = e^{-x^2}$ and $f(0) = 0$.

B) Let

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 + x + x^2 & \text{if } x \geq 0 \end{cases}$$

Potentially useful information: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

1) Using the definition of $f'(0)$, determine whether $f(x)$ is *differentiable* at $x = 0$.

2) State a general theorem that implies $f(x)$ is integrable on $[-1, 1]$.

3) *Without using the Fundamental Theorem of Calculus*, find the value of $\int_{-1}^{1} f(x)\, dx$.

C) Show that if $g : (a, b) \to \mathbf{R}$ is differentiable at some $c \in (a, b)$ with $g'(c) \neq 0$, then there is a $\delta > 0$ such that $g(x) \neq g(c)$ for all x with $0 < |x - c| < \delta$. (Hint: Argue by contradiction, and "think sequentially.")

D)

1) Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0$, $g'(b) > 0$. Show that there exists some $x_1 \in (a, b)$ where $g(x_1) < g(a)$ and also some $x_2 \in (a, b)$ such that $g(x_2) < g(b)$.

2) Deduce from part 1 that there is some $c \in (x_1, x_2)$ where $g'(c) = 0$. (Hint: "think Rolle.")

E) Let f be differentiable on (a, b) and continuous on $[a, b]$. Assume also that and $|f'(x)| < 1$ for all $x \in (a, b)$. Show that there exists $0 \leq c \leq 1$ such that

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in [a, b]$.

F)

1) Let $f(x) = 2x^2 + 3x + 3$. Show directly (i.e. using the definition via upper and lower sums) that f is integrable on $[0, 1]$, and determine the value of $\int_0^1 2x^2 + 3x + 3 \; dx$.

2) Let

$$g(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } 1 \leq x \leq 2 \end{cases}$$

Is g integrable on $[0, 2]$? Why or why not? If so, determine the value $\int_0^2 g(x) \; dx$.

G) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x^2 + 3x + 3$.

1) Show that f is uniformly continuous on $[0, 1]$ directly from the definition.

2) Is there a general theorem that gives the same result as part 1? Explain.

3) Is f uniformly continuous on $\mathbf{R}$? Prove your assertion.