Mathematics 242 – Principles of Analysis
Information on Exam 1
February 18, 2011

General Information

The first exam for the course will be given in class on Friday, February 25. This will be an individual, in-class, closed book exam. I will be happy to hold a late afternoon or evening review session to help you prepare. However, I will be leaving campus about 4:00pm on Wednesday, February 23. So Tuesday or Thursday would probably be our best choices.

What to Expect

The exam will have four or five questions, each possibly with several parts. Some questions will ask for a precise statement of a definition or a theorem we have discussed. I might ask you to give one of the proofs of key results we have covered. The Sample Questions below should give you some idea what they might look like. (The actual exam questions will be different of course!)

Topics to be Covered

The exam will cover the material we have covered since the start of the semester up to and including class on Friday, February 18. This includes all of the material from Chapter 1 in the text, the ideas about finite, countably infinite, and uncountable sets from February 16 and 18, and:

1) Sets, set operations (unions, intersections, complements)
2) Functions, the one-to-one and onto properties
3) The Well Ordering Property of **N** and the technique of proof by mathematical induction. Application to the binomial theorem.
4) Algebraic and order properties of **R**. Know the proof that $\sqrt{2} \notin \mathbf{Q}$.
5) Upper and lower bounds, least upper bound and greatest lower bound. The Axiom of Completeness for the real number system: Know the statement and what it means. Be able to give the proof of the result called Lemma 1.3.7 in the text.
6) Consequences of Completeness (know these proofs):
   (a) **N** is not bounded in **R**.
   (b) If $\varepsilon > 0$, then there exists $n \in \mathbf{N}$ such that $0 < \frac{1}{n} < \varepsilon$.
   (c) If $a < b$ in **R**, then there are infinitely many distinct rationals $r \in \mathbf{Q}$ such that $a < r < b$ and also infinitely many irrationals $s \in \mathbf{Q}^c$ such that $a < s < b$.
5) Countable and uncountable sets: Know the definition, and be able to show that the set of all (positive and negative) integers **Z** is a countable set.

Practice Questions

Don't be too concerned about the length of this list. The actual exam will be roughly half the total length of these questions. The idea is to show the range of different types of questions and topics that might be covered.

I. Let $A = [1, 5)$, $B = (-1, 2]$.

A) What are $A \cup B$, $A \cap B$, $A - B$?
B) What are lub(A), lub(B)?
C) Let $C = \{x \cdot y | x \in A, y \in B\}$. Identify lub($C$) and prove your claim.
D) Let $D = \{1/x | x \in A\}$. What are lub(D) and glb(D)? Give an explanation how you know your claims are true.
E) Prove that if E is any bounded set of strictly positive real numbers, and $F = \{1/x | x \in E\}$, then lub(F) $= 1/$glb(E).

II.
A) Prove by induction: $2n^3 + 3n^2 + n$ is divisible by 6 for all $n \geq 1$.
B) Prove by induction: $n! > 2^n$ for all natural numbers $n \geq 4$.

III. Saying a function $f : \mathbf{R} \to \mathbf{R}$ is strictly increasing means: for all real numbers x and y satisfying $x < y$, the inequality $f(x) < f(y)$ holds.

A) Give the *negation* of the statement f is strictly increasing in the same sort of form as the definition above. (What does it mean for that statement to be false. Just saying "f is not strictly increasing" is not an acceptable answer!)
B) Give the contrapositive of the statement: "If f is one-to-one, then f is strictly increasing." Is that a true statement?
C) Consider $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$. What is wrong with the following reasoning? If $x < y$, then $x^2 < y^2$; therefore $f(x) = x^2$ is a strictly increasing function.

IV.
A) State the Axiom of Completeness for the real number system.
B) Show that for any real number number $y > 1$, there exists a natural number n such that $1 + \frac{1}{n} < y$.
C) Let n represent a natural number and let $A = \cap_{n=1}^{\infty} \left[1, 1 + \frac{1}{n}\right]$. Is A the empty set? Why or why not?
D) Let $B = \cup_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right)$. What is lub($B$)? Explain.

V.
A) Prove that if $a < b$ are any real numbers, then there is some $r \in \mathbf{Q}$ such that $a < r < b$.
B) Show that if r is a rational number, then $\sqrt{2} + r$ is an irrational number. (Argue by contradiction.)

C) Show that if $a < b$ are any real numbers, then there is an irrational number s with $a < s < b$.

VI.

A) What does it mean for a set to be countably infinite?

B) Show that $\mathbf{Z}$ is countably infinite.

C) Is there a one-to-one and onto function $f : \mathbf{R} \to (-1, 1)$? If so find one; if not say why not.