Mathematics 242–Principles of Analysis
Solutions for Practice Questions for Exam 2
October 31, 2005

I.

A) This is impossible by the Order Limit Theorem (2.3.4). If $a_n < 0$ for all n and (a_n) converges to b then $b \leq 0$.

B) Example: $x_n = (-1)^n$. Note $x_n^2 = 1$ for all n.

C) Example: $x_n = n$, $y_n = \frac{1}{n^3}$. Then $\sum_n x_n y_n = \sum_n \frac{n}{n^3} = \sum_n \frac{1}{n^2}$ which converges.

D) Example: $x_n = n$ (or any other monotone increasing sequence that is not bounded above).

E) This is impossible by the result of Theorem 2.7.6 in the text.

F) Example: $\sum_{n=1}^{\infty}(-1)^n$. The partial sums are $-1, 0, -1, 0, \cdots$.

G) This is impossible because if you have any single convergent subsequence, then any subsequence *of that subsequence* is also convergent, and to the same limit as the first subsequence (Theorem 2.5.2 in the text).

II.

A) This is a geometric series with first term $-8/27$ and ratio $-2/3$. The sum is $\frac{-8/27}{1+2/3} = \frac{-8}{45}$

B) For all $n \geq 2$,

$$\frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$$

So for $m \geq 2$ the mth partial sum is

$$\begin{aligned} s_m &= \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \cdots + \frac{1}{m(m-1)} \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{m-1} - \frac{1}{m} \\ &= 1 - \frac{1}{m}. \end{aligned}$$

(This is called a "telescoping" series because of these cancellations.) Then

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{m\to\infty}\left(1 - \frac{1}{m}\right) = 1.$$

III.

A) $\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ is divergent by comparison with the harmonic series: $\frac{1}{k^{2/3}} > \frac{1}{k}$ for all $k \geq 1$. For all $k \in \mathbf{N}$, we have $k^{2/3} < (k+1)^{2/3}$, so $\frac{1}{k^{2/3}} > \frac{1}{(k+1)^{2/3}}$. Moreover $\lim_{k\to\infty} \frac{1}{k^{2/3}} = 0$. Hence by the Alternating Series Test, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2/3}}$ converges. The series is *conditionally convergent.*

B) For the Ratio Test, we look at

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}9^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n 9^n}\right| \\ &= \lim_{n\to\infty}\frac{9}{(2n+2)(2n+1)} \\ &= 0\end{aligned}$$

Since this is < 1, the Ratio Test implies the series is *absolutely convergent.*

C) This series is *divergent* because $\lim_{k\to\infty}\frac{k^2-1}{k^2+2k+4} = 1 \neq 0$ (see Theorem 2.7.3 in text).

D) For all $k \in \mathbf{N}$, $k < 2^k$. Hence $\frac{k}{5^k} < \left(\frac{2}{5}\right)^k$. Since $\sum_k \left(\frac{2}{5}\right)^k$ is a geometric series with ratio $0 < r = 2/5 < 1$, it is convergent. So the given series converges too by the Comparison Test.

E) Diverges by comparison with $\sum_k \frac{1}{2\sqrt{k+1}}$.

IV.

A) If (x_n) is convergent with limit a, then for all $\varepsilon > 0$, there exist $N \in \mathbf{N}$ such that $|x_n - a| < \varepsilon/2$ whenever $n \geq N$. But then for that N,

$$|x_n - x_{n+1}| = |(x_n - a) - (x_{n+1} - a)| \leq |x_n - a| + |x_{n+1} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence (x_n) is pC.

B) The converse of the statement in A is *false.* Here is a counterexample. Consider the sequence (s_n) of partial sums of the *divergent* harmonic series $\sum_k \frac{1}{k}$. (s_n) is a divergent sequence since $\lim_{n\to\infty} s_n = +\infty$. But $s_{n+1} - s_n = \frac{1}{n+1}$. So given any $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $|s_{n+1} - s_n| < \varepsilon$ whenver $n \geq N$ – just take $N \geq \frac{1}{\varepsilon}$, then $n \geq N$ implies $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

V.

A) Since $\lim_{n\to\infty} na_n = \ell \neq 0$, and $a_n > 0$, the Order Limit Theorem implies that $\ell > 0$. For all $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $|na_n - \ell| < \varepsilon$ for all $n \geq N$. Let ε be small enough that $\ell - \varepsilon > 0$. Then $na_n > \ell - \varepsilon$ whenever $n \geq N$, so $a_n > \frac{\ell-\varepsilon}{n}$ whenever $n \geq N$. This implies that $\sum_{n=N}^{\infty} a_n$ diverges, by comparison with the series $\sum_{n=N}^{\infty}(\ell - \varepsilon)\frac{1}{n}$, which is a constant times the "tail" of the harmonic series. As a consequence, the whole series $\sum_{n=1}^{\infty} a_n$ diverges too.

B) Since $\lim_{n\to\infty} n^2 a_n = \ell \neq 0$, and $a_n > 0$, the Order Limit Theorem implies that $\ell > 0$. For all $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $|n^2 a_n - \ell| < \varepsilon$ for all $n \geq N$. Then $n^2 a_n < \ell + \varepsilon$ whenever $n \geq N$, so $a_n < \frac{\ell-\varepsilon}{n^2}$ whenever $n \geq N$. This implies that $\sum_{n=N}^{\infty} a_n$ converges, by comparison with the series $\sum_{n=N}^{\infty}(\ell + \varepsilon)\frac{1}{n^2}$, which is a constant times the "tail" of the series $\sum_n \frac{1}{n^2}$. As a consequence, the whole series $\sum_{n=1}^{\infty} a_n$ converges too.

VI.

A) S is nonempty by the Bolzano-Weierstrass theorem. If $-M \leq x_n \leq M$ for all n and $s \in S$, then $s = \lim_k x_{n_k}$, for some subsequence x_{n_k}. The Order Limit Theorem implies that because $-M \leq x_{n_k} \leq M$, $-M \leq s \leq M$ too. Hence S is bounded above and below.

B) $\limsup \, (-1)^n + \frac{1}{n} = 1$ (look at the even-numbered terms). The odd-numbered terms show that $-1 \in S$ for this sequence too.

C) (by contradiction) Suppose that there exists some $\varepsilon > 0$ such that infinitely many terms of the sequence satsify $x_n > t+\varepsilon$. From these infinitely many terms, we select a subsequence x_{n_k} with $x_{n_k} > t+\varepsilon$ for all $k \in \mathbf{N}$. The (x_{n_k}) is also a bounded sequence, so the Bolzano-Weierstrass Theorem implies that it has a convergent subsequence $x_{n_{k_\ell}}$. Say that convergent subsequence has limit s. Then $s \in S$ since $x_{n_{k_\ell}}$ is also a subsequence of the original x_n. The Order Limit Theorem implies $s \geq t+\varepsilon > t$. This contradicts the definition of t as $t = \sup S$.