Mathematics 242 – Principles of Analysis
Review Sheet – Midterm Exam 3
November 21, 2005

General Information

As you know from the course syllabus, the third and final midterm exam for the course will be given in class on Friday, December 2. This be an individual closed book exam similar in format to the other midterms we have done this semester. I will be happy to hold a late afternoon or evening review session to help you prepare. Late afternoon times are possible Wednesday, November 30, but I will have to leave campus no later than 5:00pm to get to a rehearsal, so Wednesday evening will not work for me that week. Tuesday or Thursday evening would be OK.

Topics to be Covered

The exam will cover the material we have covered since the last exam, starting with the material on limits of functions, and going through the discussion of integrability from class on Monday, November 21. This is sections 4.1-4.5, 5.2, 5.3, and 7.2 in the text, but as before, not all the topics in those sections were discussed in class. You are responsible for only what we did talk about:

1) The definition of the statement $\lim_{x\to c} f(x) = L$, consequences, limit theorems, techniques for computing limits, including use of sequences to detect when limits do not exist, or to prove that they do.
2) The definition of continuity and its consequences, key properties of continuous functions on a closed interval: The Intermediate Value and Extreme Value Theorems.
3) The definition of differentiability and examples.
4) Rolle's Theorem, the Mean Value Theorem and its consequences.
5) The definition of integrability, computations of definite integrals from the definition.

What to Expect

The exam will have four or five questions, each possibly with several parts. Some questions will ask for a precise statement of a definition or a theorem we have discussed. Be prepared to give careful statements of the definitions noted above and know how to use them (for instance how to show that a limit exists using the ε, δ definition). Also know and be able to give these proofs:

1) The Extreme Value Theorem.
2) The Mean Value Theorem.
3) A monotone increasing function on an interval $[a, b]$ is integrable.

The other questions will be similar to questions from the problem sets and discussions.

(over for review problems)

A) Prove the following statements using the $\varepsilon - \delta$ definition of functional limits:

1) $\lim_{x\to 0} x^2 - 2x + 4 = 4$.
2) $\lim_{x\to e}[[x]] = 2$ ($[[x]]$ is the greatest integer function)
3) $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

B)

1) Prove using a sequential technique that $\lim_{x\to 0}\cos(1/x)$ *does not exist.*
2) Assume that $f, g : A \to \mathbf{R}$ and f is bounded on A (that is, there exists M such that $|f(x)| \leq M$ for all $x \in A$. Show using the $\varepsilon - \delta$ definition that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} f(x)g(x) = 0$ also.
3) Use part 2 to show that $\lim_{x\to 0} x^a \cos(1/x) = 0$ for all $a > 0$.

C) For each of the following, give an example or a short proof that no such examples exist:

1) A function $g : \mathbf{R} \to \mathbf{R}$ that is continuous at $x = 0$ but at no other $x \in \mathbf{R}$.
2) A function $f : \mathbf{R} \to \mathbf{R}$ that is continuous at $x \in \mathbf{N}$ but at no other $x \in \mathbf{R}$.
3) A function $f : [0,1] \to \mathbf{R}$ such that $3 = \sup\{f(x) : x \in [0,1]\}$, but there is no $x \in [0,1]$ with $f(x) = 3$.
4) A continuous function $f : [0,1] \to \mathbf{R}$ such that $3 = \sup\{f(x) : x \in [0,1]\}$, but there is no $x \in [0,1]$ with $f(x) = 3$.
5) A continuous function $f : [0,1] \to [0,1]$ with $f(0) = 0$, $f(1) = 1$, but such that there is no $x \in [0,1]$ with $f(x) = 1/3$.
6) A continuous function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = 0$ if $x \in \mathbf{Q}$, but $f(\pi) = 3$.
7) A function $f : [a,b] \to \mathbf{R}$ that is differentiable on $[a,b]$, satisfies $f(a) = f((a+b)/2) = f(b)$, but has only one critical point in (a,b). (Recall, a critical point is a c where $f'(c) = 0$ or $f'(c)$ does not exist.)

D) Show that if $g : (a,b) \to \mathbf{R}$ is differentiable at some $c \in (a,b)$ with $g'(c) \neq 0$, then there is a $\delta > 0$ such that $g(x) \neq g(c)$ for all x with $0 < |x - c| < \delta$. (Hint: Argue by contradiction, and "think sequentially.")

E)

1) Assume that g is differentiable on $[a,b]$ and satisfies $g'(a) < 0$, $g'(b) > 0$. Show that there exists some $x_1 \in (a,b)$ where $g(x_1) < g(a)$ and also some $x_2 \in (a,b)$ such that $g(x_2) < g(b)$.
2) Deduce from part 1 that there is some $c \in (x_1, x_2)$ where $g'(c) = 0$. (Hint: "think Rolle")

   Note: this argument is the key part of the proof of an interesting result about derivatives called "Darboux's Theorem" – if g is differentiable on $[a,b]$, then $g'(x)$ has the *intermediate value property* (that is, for all α between $g'(a)$ and $g'(b)$, there is some $cin(a,b)$ where $g'(c) = \alpha$).

F) Show using the MVT that if f is differentiable on $[a, b]$, f' is continuous on that interval, and $|f'(x)| < 1$ for all $x \in [a, b]$, then f satisfies the hypothesis of the Contraction Mapping Theorem on $[a, b]$: $|f(x) - f(y)| \leq c|x - y|$ for some $0 \leq c \leq 1$.

G)

1) Let $f(x) = 2x^2 + 3x + 3$. Show directly (i.e. using the definition via upper and lower sums) that f is integrable on $[0, 1]$, and determine the value of $\int_0^1 2x^2 + 3x + 3 \, dx$.
2) Let

$$g(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } 1 \leq x \leq 2 \end{cases}$$

Is g integrable on $[0, 2]$? Why or why not? If so, determine the value $\int_0^2 g(x) \, dx$.