Mathematics 242 - Principles of Analysis
Solutions - Exam 2
November 4, 2005

I.

A) The Monotone Convergence Theorem says: Every monotone bounded sequence in $\mathbf{R}$ is convergent. Proof (in monotone increasing case): Say (x_n) is a monotone increasing sequence that is bounded above. By the Axiom of Completeness, $a = \sup\{x_n : n \in \mathbf{N}\}$ exists in $\mathbf{R}$. By a result we know about least upper bounds, for all $\varepsilon > 0$, there exists some N such that $a - \varepsilon < x_N \leq a$. But then since the sequence is monotone increasing, $a - \varepsilon < x_N \leq x_n \leq a$ for all $n \geq N$. It follows that $|x_n - a| < \varepsilon$ for all $n \geq N$, so by definition the sequence converges to a.

B) Since $x_n < 0$, we have $x_{n+1} \geq x_n$ for all n. This means that the sequence is monotone increasing and bounded above by 0. Hence part A shows that $\lim_{n\to\infty} x_n = a$ for some real number a. Furthermore $a \leq 0$ by the Order Limit Theorem since $x_n < 0$ for all n.

II.

A) The Bolzano-Weierstrass theorem says that every bounded sequence has a convergent subsequence.

B) We have $|x_n| = |\sin(n)| \leq 1$ for all n. Hence x_n is a bounded sequence and the Bolzano-Weierstrass theorem applies. There is a subsequence x_{n_k} that converges.

III.

A) $\lim_{k\to\infty} ka_k = 4$ means that for $\epsilon = 1$, for instance, there exists a natural number N such that $ka_k > 4 - 1 = 3$ for all $k > N$. Then $a_k > \frac{3}{k}$ for these k, and the series $\sum_k a_k$ diverges by comparison with the series $\sum_k \frac{3}{k}$. So *there are no such examples.*

B) Any convergent series of negative terms is an example, for instance $\sum_{k=0}^{\infty} \frac{-1}{2^k}$.

IV.

A) $\sum_{k=1}^{\infty} \frac{1}{5k+3}$ diverges by the comparison test: For all $k > 3$: $5k+3 < 6k$, so $\frac{1}{5k+3} > \frac{1}{6k}$. The series $\sum_{k=1}^{\infty} \frac{1}{6k}$ diverges because it is a constant times the harmonic series. However $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{5k+3}$ converges by the Alternating Series Test since $\frac{1}{5k+3}$ is strictly decreasing as $k \to \infty$ and $\lim_{k\to\infty} \frac{1}{5k+3} = 0$. The given series is *conditionally convergent.*

B) This is a geometric series with ratio $\frac{-1}{7}$. Since $|\frac{-1}{7}| < 1$ it is *absolutely convergent.*

V. Using the Ratio Test,

$$\lim_{k\to\infty} \left| \frac{(x-3)^{k+1}}{(k+1)6^{k+1}} \cdot \frac{k6^k}{(x-3)^k} \right| = \lim_{k\to\infty} |x-3| \left(\frac{k+1}{6k} \right) = \frac{|x-3|}{6}$$

This is < 1 when $|x-3| < 6$, or $-3 < x < 9$. The series converges absolutely for these x and diverges for $x > 9$ and $x < -3$. When $x = 9$ we get the harmonic series which diverges. When $x = -3$, then we get the alternating harmonic series which converges by the Alternating Series Test. Hence the whole interval of convergence is $-3 \le x < 9$.

Extra Credit. Let $\varepsilon > 0$. Since x_n converges to a there exists a natural number N_1 such that $|x_n - a| < \varepsilon/2$ whenever $n \ge N_1$. So when $n \ge N_1$, we have by the triangle inequality

$$\begin{aligned}
|y_n - a| &= \left| \frac{x_1 + x_2 + \cdots + x_n - na}{n} \right| \\
&\le \frac{1}{n} \left(|x_1 - a| + \cdots + |x_{N_1 - 1} - a| + |x_{N_1} - a| + \cdots + |x_n - a| \right) \\
&\le \frac{1}{n} \left(|x_1 - a| + \cdots + |x_{N_1 - 1} - a| \right) + \frac{n - N_1 + 1}{n} \varepsilon/2 \\
&< \frac{1}{n} \left(|x_1 - a| + \cdots + |x_{N_1 - 1} - a| \right) + \varepsilon/2
\end{aligned}$$

Now, using the Hint, notice that for all k, there exists a real number M such that $|x_k - a| \le M$ (a convergent sequence is bounded). Therefore, the last term above is bounded above by $\frac{N_1 - 1}{n} M$. Given ε, N_1 is fixed as above. But if we take n sufficiently large, say $n > N_2$ for some N_2, then we can make $\frac{N_1 - 1}{n} M < \varepsilon/2$ also. Therefore $|y_n - a| < \varepsilon$ for all $n > \max(N_1, N_2)$.