I.
A) The Monotone Convergence Theorem says: Every monotone bounded sequence in \( R \) is convergent. Proof (in monotone increasing case): Say \((x_n)\) is a monotone increasing sequence that is bounded above. By the Axiom of Completeness, \( a = \sup\{x_n : n \in \mathbb{N}\} \) exists in \( R \). By a result we know about least upper bounds, for all \( \varepsilon > 0 \), there exists some \( N \) such that \( a - \varepsilon < x_N \leq a \). But then since the sequence is monotone increasing, \( a - \varepsilon < x_N \leq x_n \leq a \) for all \( n \geq N \). It follows that \( |x_n - a| < \varepsilon \) for all \( n \geq N \), so by definition the sequence converges to \( a \).

B) Since \( x_n < 0 \), we have \( x_{n+1} \geq x_n \) for all \( n \). This means that the sequence is monotone increasing and bounded above by \( 0 \). Hence part A shows that \( \lim_{n \to \infty} x_n = a \) for some real number \( a \). Furthermore \( a \leq 0 \) by the Order Limit Theorem since \( x_n < 0 \) for all \( n \).

II.
A) The Bolzano-Weierstrass theorem says that every bounded sequence has a convergent subsequence.

B) We have \( |x_n| = |\sin(n)| \leq 1 \) for all \( n \). Hence \( x_n \) is a bounded sequence and the Bolzano-Weierstrass theorem applies. There is a subsequence \( x_{n_k} \) that converges.

III.
A) \( \lim_{k \to \infty} ka_k = 4 \) means that for \( \varepsilon = 1 \), for instance, there exists a natural number \( N \) such that \( ka_k > 4 - 1 = 3 \) for all \( k > N \). Then \( a_k > \frac{3}{k} \) for these \( k \), and the series \( \sum_k a_k \) diverges by comparison with the series \( \sum_k \frac{3}{k} \). So there are no such examples.

B) Any convergent series of negative terms is an example, for instance \( \sum_{k=0}^{\infty} \frac{1}{2^k} \).

IV.
A) \( \sum_{k=1}^{\infty} \frac{1}{5k+3} \) diverges by the comparison test: For all \( k > 3: 5k+3 < 6k \), so \( \frac{1}{5k+3} > \frac{1}{6k} \). The series \( \sum_{k=1}^{\infty} \frac{1}{6k} \) diverges because it is a constant times the harmonic series. However \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{5k+3} \) converges by the Alternating Series Test since \( \frac{1}{5k+3} \) is strictly decreasing as \( k \to \infty \) and \( \lim_{k \to \infty} \frac{1}{5k+3} = 0 \). The given series is conditionally convergent.

B) This is a geometric series with ratio \( \frac{1}{7} \). Since \( |\frac{1}{7}| < 1 \) it is absolutely convergent.

V. Using the Ratio Test,

\[
\lim_{k \to \infty} \left| \frac{(x - 3)^{k+1}}{(k+1)6^{k+1}} \cdot \frac{k6^k}{(x - 3)^k} \right| = \lim_{k \to \infty} \left| \frac{x - 3}{6} \right| \left( \frac{k+1}{k} \right) = \left| \frac{x - 3}{6} \right|
\]

This is < 1 when \( |x - 3| < 6 \), or \(-3 < x < 9 \). The series converges absolutely for these \( x \) and diverges for \( x > 9 \) and \( x < -3 \). When \( x = 9 \) we get the harmonic series which
diverges. When \( x = -3 \), then we get the alternating harmonic series which converges by the Alternating Series Test. Hence the whole interval of convergence is \(-3 \leq x < 9\).

**Extra Credit.** Let \( \varepsilon > 0 \). Since \( x_n \) converges to \( a \) there exists a natural number \( N_1 \) such that \( |x_n - a| < \varepsilon/2 \) whenever \( n \geq N_1 \). So when \( n \geq N_1 \), we have by the triangle inequality

\[
|y_n - a| = \left| \frac{x_1 + x_2 + \cdots + x_n - na}{n} \right|
\]

\[
\leq \frac{1}{n} \left( |x_1 - a| + \cdots + |x_{N_1-1} - a| + |x_{N_1} - a| + \cdots + |x_n - a| \right)
\]

\[
\leq \frac{1}{n} \left( |x_1 - a| + \cdots + |x_{N_1-1} - a| \right) + \frac{n - N_1 + 1}{n} \varepsilon/2
\]

\[
< \frac{1}{n} \left( |x_1 - a| + \cdots + |x_{N_1-1} - a| \right) + \varepsilon/2
\]

Now, using the Hint, notice that for all \( k \), there exists a real number \( M \) such that \( |x_k - a| \leq M \) (a convergent sequence is bounded). Therefore, the last term above is bounded above by \( \frac{N_1-1}{n} M \). Given \( \varepsilon \), \( N_1 \) is fixed as above. But if we take \( n \) sufficiently large, say \( n > N_2 \) for some \( N_2 \), then we can make \( \frac{N_1-1}{n} M < \varepsilon/2 \) also. Therefore \( |y_n - a| < \varepsilon \) for all \( n > \text{max}(N_1, N_2) \).