Mathematics 242 – Principles of Analysis
Information on Exam 2
October 28, 2005

General Information

The second hour exam for the course will be given in class on Friday, November 4. This will be an individual, in-class, closed book exam. I will be happy to hold a late afternoon or evening review session to help you prepare. Late afternoon or evening times are possible Wednesday, November 2.

Topics to be Covered

The exam will cover the material we have covered since the last exam. (Of course, all the material about logic and proofs, including the technique of proof by mathematical induction, is relevant here too!)

1) The Algebraic and Order Limit Theorems for sequences.
2) The Monotone Convergence Theorem for sequences and its consequences
3) Subsequences and the Bolzano-Weierstrass Theorem.
4) Convergence of Infinite series (the definition, key examples such as geometric series, $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$, and so forth).
5) The Comparison, Alternating Series, and Ratio tests.

What to Expect

The format will be similar to that of the first exam. The exam will have four or five questions, each possibly with several parts. Some questions will ask for a precise statement of a definition or a theorem we have discussed. Be prepared to give careful statements of

1) The definition of convergence for a sequence, and convergence for an infinite series.
2) The Bolzano-Weierstrass Theorem.

Also know and be able to give these proofs:

1) Part ii of the Algebraic Limit Theorem for sequences (2.3.3) (limit of a sum is the sum of the limits).
2) The Monotone Convergence Theorem for sequences.
3) The Comparison Test for infinite series with positive terms.

Practice Questions

I. Give an example of each of the following, or give a short proof that there are no such examples:

A) A convergent sequence with all strictly negative terms whose limit is 1.
B) A sequence (x_n) such that (x_n^2) converges but (x_n) does not.

C) Sequences (x_n) and (y_n) such that $\lim x_n = +\infty$, but $\sum_{n=1}^{\infty} x_n y_n$ converges.
D) A monotone increasing sequence (x_n) that has no convergent subsequence.
E) An absolutely convergent series which is not convergent.
F) A divergent series whose partial sums are all in a bounded interval $[-M, M]$.
G) A sequence that has *only one* convergent subsequence.

II. Find the sum of each of the following infinite series:
A) $\sum_{n=3}^{\infty} \left(\frac{-2}{3}\right)^n$
B) $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ (Hint: What are A, B to make $\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$?)

III. Say whether each of the following series converges absolutely, converges conditionally, or diverges and justify your answer:
A)

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2/3}}$$

B)

$$\sum_{k=0}^{\infty} \frac{(-1)^k 9^k}{(2k)!}$$

(Hint: the Ratio Test is good for ones like this!)
C)

$$\sum_{k=0}^{\infty} \frac{k^2 - 1}{k^2 + 2k + 4}$$

D)

$$\sum_{k=1}^{\infty} \frac{k}{5^k}$$

E)

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + \sqrt{k+1}}$$

IV. In this problem, we say a sequence (x_n) is pC ("pseudo-Cauchy") if for every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $|x_n - x_{n+1}| < \varepsilon$ for all $n \geq N$ (the "real Cauchy" sequences are the ones where $|x_n - x_m| < \varepsilon$ for all $m, n \geq N$–be sure you see the difference!)
A) Show that if (x_n) is convergent, then (x_n) is pC.
B) Is the converse of the statement in part A true? Why or why not? (Hint: Think about sequences of partial sums of infinite series.)

V. "Limit-comparison" tests.
A) Show that if $a_n > 0$ and $\lim_{n\to\infty} n a_n = \ell$ with $\ell \neq 0$, then $\sum_n a_n$ *diverges.*
B) Show that if $a_n > 0$ and $\lim_{n\to\infty} n^2 a_n = \ell$ with $\ell \neq 0$, then $\sum_n a_n$ *converges.*

VI. In all parts of this problem, (x_n) is a bounded sequence. Let S be the set of all real numbers s such that there exists a subsequence (x_{n_k}) converging to s.

A) Show that S is bounded above and below.

The number $t = \sup S$ is sometimes called the *limit superior* of the sequence (x_n), written

$$t = \limsup\ x_n.$$

B) What is $\limsup\ (-1)^n + \frac{1}{n}$?

C) Show that if $\varepsilon > 0$ and $t = \limsup\ x_n$, then there are only finitely many terms of the sequence x_n with $x_n > t + \varepsilon$.