Mathematics 242 – Principles of Analysis
Solutions for Exam 1 – October 7, 2005

I. A) (10) State the Axiom of Completeness for the real number system.

Solution: Every set $A \subset \mathbf{R}$ that is bounded above has a least upper bound $s = \sup(A)$ in $\mathbf{R}$.

B) (10) Show that if $s = \sup(A)$ for some $A \subset \mathbf{R}$ and $\varepsilon > 0$, then there exists some $a \in A$ satisfying $s - \varepsilon < a \leq s$.

Solution: By definition, s is the real smallest number that is an upper bound for A. Since $\varepsilon > 0$, $s - \varepsilon < 0$ and hence $s - \varepsilon$ is *not* an upper bound for A. This implies that there is some $a \in A$ with $s - \varepsilon < a$. $a \leq s$ because s is an upper bound for A.

C) (10) Let $A = \cup_{n=1}^{\infty} \left(0, 1 - \frac{1}{2n}\right)$. Explain why A is bounded above and determine $\sup(A)$.

Solution: If $x \in A$, then there is some n such that $x < \frac{1}{2n}$. Since $\frac{1}{2n} < 1$ for all $n \in \mathbf{N}$, we get $x < 1$ for all $x \in A$. In fact, $\sup(A) = 1$ since if $y = 1 - \varepsilon < 1$, then the Archimedean Property implies that there exists $n \in \mathbf{N}$ such that $\frac{1}{2n} < \varepsilon$, so $1 - \frac{1}{2n} > 1 - \varepsilon = y$. Hence there exist $x \in A$ with $x > y$ too, and y cannot be an upper bound for A. Since 1 is an upper bound for A, but no $y < 1$ is, $\sup(A) = 1$.

II. A) (5) Find

$$\lim_{n\to\infty} \frac{2n^2 - 1}{5n^2 + 2n}$$

using the Algebraic Limit Theorem.

Solution:

$$\lim_{n\to\infty} \frac{2n^2 - 1}{5n^2 + 2n} = \lim_{n\to\infty} \frac{2 - \frac{1}{n^2}}{5 + \frac{2}{n}} = \frac{2}{5}$$

B) (15) Prove that $x_n = \frac{2n^2-1}{5n^2+2n}$ converges to your limit from part A *using the definition.*

Solution: Preliminaries: We can use some slightly tricky estimates to reduce the complication of the N that works for a given $\varepsilon > 0$:

$$\begin{aligned}\left|\frac{2n^2 - 1}{5n^2 + 2n} - \frac{2}{5}\right| &= \frac{4n + 5}{25n^2 + 10n} \\ &< \frac{4n + 5}{25n^2} \\ &< \frac{5n + 5}{25n^2} \\ &= \frac{n + 1}{5n^2} \\ &< \frac{2n}{5n^2} = \frac{2}{5n}\end{aligned}$$

Proof: Let $\varepsilon > 0$, and let $N \in \mathbf{N}$ satisfy $N > \frac{2}{5\varepsilon}$. Then for all $n \geq N$,

$$\begin{aligned}
\left|\frac{2n^2-1}{5n^2+2n} - \frac{2}{5}\right| &= \frac{4n+5}{25n^2+10n} \\
&< \frac{4n+5}{25n^2} \\
&< \frac{5n+5}{25n^2} \\
&= \frac{n+1}{5n^2} \\
&< \frac{2n}{5n^2} \\
&\leq \frac{2}{5N} \\
&< \varepsilon
\end{aligned}$$

Hence $\lim_{n\to\infty} \frac{2n^2-1}{5n^2+2n} = \frac{2}{5}$.

III. Let (x_n) be the sequence defined by $x_1 = 2$ and $x_{n+1} = \frac{2x_n - 1}{3}$ for all $n \geq 1$.

A) (15) Using mathematical induction, show that $x_n \geq -1$ for all $n \geq 1$.

Solution: The base case is $n = 1$, and $x_1 = 2 \geq -1$ from the definition of the sequence. For the induction step, assume $x_k \geq -1$, and apply the operations on both sides to produce x_{k+1} on the left:

$$x_{k+1} = \frac{2x_k - 1}{3} \geq \frac{2(-1)-1}{3} = -1.$$

Hence by induction, $x_n \geq -1$ for all $n \in \mathbf{N}$.

B) (15) Using mathematical induction, show that (x_n) is strictly decreasing: $x_{n+1} < x_n$ for all $n \geq 1$.

Solution: With $n = 1$, we have $x_2 = \frac{2(2)-1}{3} = 1 < x_1 = 2$. Now assume that $x_{k+1} < x_k$ and apply the operations producing x_k from x_{k+1} to both sides:

$$x_{k+2} = \frac{2x_{k+1} - 1}{3} < \frac{2x_k - 1}{3} = x_{k+1}.$$

This shows $x_{n+1} < x_n$ for all $n \in \mathbf{N}$.

IV. True-False. For each true statement give a short proof or reason; for each false statement give a counterexample.

A) (10) If A and B are sets of real numbers with the property that $a > b$ for all $a \in A$ and $b \in B$, then $\inf(A) > \sup(B)$.

Solution: This is FALSE. It can be true that $a > b$ for all $a \in A$ and $b \in B$, but $\inf(A) = \sup(B)$. Counterexample to the statement as given: $A = \left\{1 + \frac{1}{n} : n \in \mathbf{N}\right\}$ and $B = \left\{1 + \frac{1}{n} : n \in \mathbf{N}\right\}$. All elements of A are strictly bigger than 1, while all elements of B are strictly less than 1. Hence $a > b$ for all $a \in A$ and $b \in B$. But $\inf(A) = \sup(B) = 1$.

B) (10) If r is a nonzero rational number and t is an irrational number, then rt is irrational.

Solution: This is TRUE. Arguing by contradiction, suppose $rt \in \mathbf{Q}$. Write $r = a/b, rt = c/d$ with $a, b, c, d \in \mathbf{Z}$. Then $at/b = c/d$, or $t = (bc)/(ad) \in \mathbf{Q}$. This is a contradiction. so t must be irrational.