Mathematics 242 – Principles of Analysis
Information on Exam 1
September 28, 2005

General Information

The first exam for the course will be given in class on Wednesday, October 5. This will be an individual, in-class, closed book exam. I will be happy to hold a late afternoon or evening review session to help you prepare. Monday, October 3 will probably be the best day for that.

What to Expect

The exam will have four or five questions, each possibly with several parts. Some questions will ask for a precise statement of a definition or a theorem we have discussed. I might ask you to give one of the proofs of key results we have covered. The Sample Questions below should give you some idea what they might look like. (The actual exam questions will be different of course!)

Topics to be Covered

The exam will cover the material we have covered since the start of the semester up to and including class on September 28. This includes material from Chapter 1 and sections 2 and 3 of Chapter 2 in the text. Chapter 1 also contains some topics we did not discuss in class; you *will not be responsible for those*.

1) Logic and proofs: Know how to form the converse and contrapositive of an “if-then” statement, know how to negate statements containing the quantifiers “for all” and “there exists”, understand the idea of proof by contradiction and proofs by mathematical induction
2) Upper and lower bounds, least upper bound and greatest lower bound. The Axiom of Completeness for the real number system: Know the statement and what it means. Be able to give the proof of the result called Lemma 1.3.7 in the text.
3) Consequences of Completeness: The Nested Interval Property, the Archimedean Property, and the Density of $\mathbf{Q}$ in $\mathbf{R}$. Be able to state these results and be prepared to apply them to deduce consequences (see Practice Question III below for an example).
4) Countable and uncountable sets: Know the definition, and be able to show that the set of all (positive and negative) integers $\mathbf{Z}$ is a countable set.
5) Limits of sequences: Know the definition of convergence and be able to use it to prove that reasonably simple sequences converge to a limit (examples comparable to Exercise 2.2.1 or Practice Questions V, VI below)
6) The Algebraic and Order Limit Theorems. Be able to give the proof of part 2 of the Algebraic Limit Theorem (limit of a sum is the sum of the limits).

I. Saying a function $f : \mathbf{R} \to \mathbf{R}$ is strictly increasing means: for all real numbers x and y satisfying $x < y$, the inequality $f(x) < f(y)$ holds.

A) Give the negation of the statement f is strictly increasing ("f is not strictly increasing" is not an acceptable answer!)
B) Give the converse and contrapositive of the statement: "If f is one-to-one, then f is strictly increasing." Which of these are true statements?
C) Consider $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$. What is wrong with the following reasoning? If $x < y$, then $x^2 < y^2$; therefore $f(x) = x^2$ is a strictly increasing function.

II.
A) State the Axiom of Completeness for the real number system.
B) Use the Archimedean Property to show that for any real number $y > 1$, there exists a natural number n such that $1 + \frac{1}{n} < y$.
C) Let n represent a natural number and let $A = \cap_{n=1}^{\infty} \left[1, 1 + \frac{1}{n}\right]$. Is A the empty set? Why or why not?
D) Let $B = \cap_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right)$. What is $\sup(B)$? Explain.

III.
A) What is the precise statement of the Density of $\mathbf{Q}$ in $\mathbf{R}$?
B) Show that if r is a rational number, then $\sqrt{2} + r$ is an irrational number. (Argue by contradiction.)
C) Show that if $a < b$ are any real numbers, then there is an irrational number t with $a < t < b$. Hint: What does the Density theorem imply about the interval $(a - \sqrt{2}, b - \sqrt{2})$?

IV.
A) What does it mean for a set to be countable?
B) Show that $\mathbf{Z}$ is countable.
C) Do $\mathbf{R}$ and the open interval $(0, 1)$ have the same cardinality? (Is there a one-to-one and onto function $f : \mathbf{R} \to (-1, 1)$? If so find one; if not say why not.)

V. Show using the definition that the sequence $x_n = \frac{n+1}{3n+2}$ converges to $\frac{1}{3}$.

VI. Let (x_n) be the sequence defined by $x_1 = 1$ and $x_n = \frac{3x_{n-1}+1}{5}$ for all $n \geq 2$.
A) Using mathematical induction, show that $x_n \geq 1/2$ for all $n \geq 1$.
B) Using mathematical induction, show that $\left|x_n - \frac{1}{2}\right| = \frac{1}{2} \cdot \left(\frac{3}{5}\right)^{n-1}$ for all $n \geq 1$.
C) Use the definition and part B to show that the sequence x_n converges to $1/2$.

VII.
A) Show that if (x_n) converges to x and (y_n) converges to y, then $(x_n + y_n)$ converges to $x + y$.
B) Find $\lim_{n\to\infty} \sqrt{n}(\sqrt{9n+1} - 3\sqrt{n})$.