Mathematics 242 – Principles of Analysis
Solutions – Discussion 3
November 29, 2005

A) For the regular partition P_n of $[0,2]$ with n smaller intervals we have $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$. The function $f(x) = 1 - 6x^2$ is decreasing on $[0,2]$ (since $f'(x) = -12x \leq 0$ on the interval). Hence

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} \left(1 - 6\left(\frac{2(i-1)}{n}\right)^2\right)\frac{2}{n} \\ &= \frac{2}{n}\sum_{i=1}^{n} 1 - \frac{48}{n^3}\sum_{i=1}^{n-1} i^2 \\ &= 2 - \frac{48}{n^3}\sum_{i=1}^{n-1} i^2 \end{aligned}$$

Similarly

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} \left(1 - 6\left(\frac{2i}{n}\right)^2\right)\frac{2}{n} \\ &= 2 - \frac{48}{n^3}\sum_{i=1}^{n} i^2 \end{aligned}$$

Hence,

$$U(f, P_n) - L(f, P_n) = \frac{48}{n^3}n^2 = \frac{48}{n}$$

If we are given $\varepsilon > 0$, then $\frac{48}{n} < \varepsilon$ whenever $n > \frac{48}{\varepsilon}$. So we have that f is integrable. The value of the integral is found by taking

$$\begin{aligned} \lim_{n\to\infty} L(f, P_n) &= \lim_{n\to\infty}\left(2 - \frac{48}{n^3}\sum_{i=1}^{n} i^2\right) \\ &= \lim_{n\to\infty}\left(2 - \frac{48}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\ &= 2 - 16 = -14 \end{aligned}$$

B) The answer is NO: $f(x)$ is not integrable on $[0,1]$ (or on any other interval $[a,b]$ with $b > a$). The reason is that since every such interval contains both rational and irrational numbers. So for every partition P, we have $M_i = 1$ all i, and $m_i = 0$ all i. Hence

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1,$$

while

$$L(f,P)=\sum_{i=1}^{n} m_i\Delta x_i = 0.$$

Given $\varepsilon < 1$, there is no partition P such that $U(f,P)-L(f,P)<\varepsilon$. Hence f is not integrable.

C) Consider the regular partition P_n of $[a,b]$ into n equal subintervals: $P_n=\{x_i : 0\le i\le n\}$ with $x_i - x_{i-1} = (b-a)/n$ for all i. Then since f is increasing $M_i = f(x_i)$ for all i and

$$U(f,P_n)=\frac{(b-a)}{n}\sum_{i=1}^{n} f(x_i)$$

Similarly, $m_i = f(x_{i-1})$ for each i and

$$L(f,P_n)=\frac{(b-a)}{n}\sum_{i=1}^{n} f(x_{i-1}) = \frac{(b-a)}{n}\sum_{i=0}^{n-1} f(x_i).$$

In the difference $U(f,P_n)-L(f,P_n)$, all the terms except the last one in U and the first one in L cancel (i.e. the sum "telescopes"), leaving

$$U(f,P_n)-L(f,P_n) = (b-a)(f(b)-f(a))\cdot\frac{1}{n}$$

(note $x_n = b$ and $x_0 = a$). Since $(b-a)(f(b)-f(a))$ is constant, we have $\lim_{n\to\infty} U(f,P_n) - L(f,P_n) = 0$. Therefore by Theorem 7.2.8, f is integrable on $[a,b]$.

D) Let $\varepsilon > 0$. Since f is integrable on $[a,c]$, there is a partition P_1 of $[a,c]$ such that $U(f,P_1)-L(f,P_1)<\varepsilon/2$. Similarly, since f is integrable on $[c,b]$ there is a partition P_2 of $[c,b]$ such that $U(f,P_2)-L(f,P_2)<\varepsilon/2$. Let $P_\varepsilon = P_1\cup P_2$. This gives a partition of $[a,b]$. From the definitions, $U(f,P_\varepsilon)=U(f,P_1)+U(f,P_2)$ and $L(f,P_\varepsilon)=L(f,P_1)+L(f,P_2)$. This shows that

$$U(f,P_\varepsilon)-L(f,P_\varepsilon) = (U(f,P_1)-L(f,P_1)) + (U(f,P_2)-L(f,P_2)) < \varepsilon/2+\varepsilon/2 = \varepsilon$$

Therefore, f is integrable on $[a,b]$. To see the relation $\int_a^b f = \int_a^c f + \int_c^b f$, note for all $\varepsilon > 0$, there is a partition $P_\varepsilon = P_1 \cup P_2$ as above, such that

$$\begin{aligned}\left|\int_a^b f - \left(\int_a^c f + \int_c^b f\right)\right| &= \left|\int_a^b f - L(f,P_\varepsilon) - \left(\int_a^c f - L(f,P_1)\right) - \left(\int_c^b f - L(f,P_2)\right)\right| \\ &\le \left|\int_a^b f - L(f,P_\varepsilon)\right| + \left|\int_a^c f - L(f,P_1)\right| + \left|\int_c^b f - L(f,P_2)\right| \\ &< \varepsilon + \varepsilon/2 + \varepsilon/2.\end{aligned}$$

(The last inequalities follow since $L(f,P) < \int_a^b f < U(f,P)$.) Since $\varepsilon > 0$ is arbitrary, this shows $\int_a^b f - \left(\int_a^c f + \int_c^b f\right) = 0$