Mathematics 242 – Principles of Analysis
Discussion 3: Integrability
November 21, 2005

Background

On Friday, we introduced the upper and lower sums for a bounded function f on $[a, b]$ relative to a partition P:

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

We showed that every lower sum is less than or equal to every upper sum:

$$L(f, P) \leq U(f, P')$$

for all partitions P, P' of $[a, b]$, and we introduced the terminology that f is *integrable* over $[a, b]$ if

$$\sup\{L(f, P) : P \text{ all partitions of } [a, b]\} = \inf\{U(f, P) : P \text{ all partitions of } [a, b]\}.$$

(∗) Another way to say that f is integrable is to say that for any $\epsilon > 0$, there exists some partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Using this alternate form of the definition, let's consider some examples and properties of integrable functions.

Discussion Questions

A) By using the definition (∗) and considering regular partitions of $[0, 2]$, show that $f(x) = 1 - 6x^2$ is integrable over $[0, 2]$ and compute $\int_0^2 1 - 6x^2 \, dx$.

B) Is the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is not a rational number} \end{cases}$$

integrable over $[0, 1]$? Why or why not? (Recall from the first part of the course that every interval of length > 0, no matter how small, contains both rational and irrational numbers.)

C) Here's a somewhat surprising result. Suppose f is increasing on the whole interval $[a, b]$. Show that f must be integrable over $[a, b]$ by considering the difference $U(f, P_n) - L(f, P_n)$ for a regular partition P_n of $[a, b]$ and letting $n \to \infty$. (Hint: $\Delta x = \frac{b-a}{n}$ can be factored out of every term in $U(f, P_n) - L(f, P_n)$. When you do that, what happens in the sum?)

D) If $c \in (a, b)$ and f is integrable over $[a, c]$ and $[c, b]$, then show f is integrable over $[a, b]$ also, and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Assignment

Writeups due Monday, November 28, no later than 5:00pm.