Mathematics 242 – Principles of Analysis
Discussion 2 – The Cantor Set
October 24-26, 2005

Background

In this discussion, we will study some of the surprising properties of a subset of the real number system called the *Cantor set*. This was first constructed by the same mathematician Georg Cantor who we met earlier when we discussed the uncountability of $\mathbf{R}$. The Cantor set is produced by the following process:

1) Start with the closed interval $C_0 = [0, 1]$.
2) Discard the open "middle third" $(1/3, 2/3)$ interval to make

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

3) Discard the open "middle third" of each half of C_1 to make to make

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

4) Discard the open "middle third" of each of the four pieces of C_2 to make C_3, and so on.
5) Repeat this process of removing the middle thirds to produce C_n for all $n \geq 1$.
6) The Cantor set C is then

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Discussion Questions

A) Show that the sum of the lengths of all of the intervals *removed* in the process described above is 1. (Note: there are infinitely many of them, so this is an infinite series!)

B) Since the whole interval $[0, 1]$ has length 1, you might think that means that $C = \emptyset$. But in fact that is *far from true*. In fact, C is still an *uncountably infinite* set of real numbers. Here is one way to see this:

   1) We usually use base 10 expansions to represent real numbers, but any other integer base $b > 1$ would work just as well. Let's consider base $b = 3$ or "ternary" expansions. The ternary expansion of a whole number like $n = 24$ would be found like this: $24 = 2 \cdot 3^2 + 2 \cdot 3 + 0 \cdot 1$. So we would say $24 = (220)_3$. If a number has a fractional part, then that is represented by negative powers of 3. The ternary digits of a number x in $[0, 1]$ are integers $d_i = 0, 1$, or 2 appearing in a series

$$x = \frac{d_1}{3} + \frac{d_2}{3^2} + \frac{d_3}{3^3} + \cdots$$

Show that every such series (that is, for every possible choice of $d_i \in \{0, 1, 2\}$, $i \in \mathbf{N}$) converges to some real number in $[0, 1]$.

2) What is true about the ternary digit d_1 for the numbers removed in the middle half of $[0, 1]$? What is true about d_1 for the numbers in the Cantor set C? Similarly, what is true about the ternary digit d_2 for the number removed from C_1 to get C_2? What is true about d_2 for the numbers in C?
3) Let's consider the set of all sequences (x_n) where $x_n = 0$ or 2 for each $n \in \mathbf{N}$. Show that this set of sequences is an uncountable set by adapting the Cantor Diagonalization proof we used to show $[0, 1]$ is uncountable.
4) Put together parts 1,2,3 to explain why C is uncountably infinite.

C) Here is another surprising property of C. We claim that every number in the closed interval $[0, 2]$ can be obtained as $x + y$ for some $x, y \in C$. (This is true even though C actually contains *no intervals* itself!). Follow this plan:

1) First show by induction that for each $n \geq 1$, every number in $[0, 2]$ can be obtained as $x_n + y_n$ for some $x_n, y_n \in C_n$.
2) The (x_n) and (y_n) are bounded sequences. Deduce the desired statement by applying Bolzano-Weierstrass (and other results we have seen, as needed!)

Assignment

Writeups due Friday, October 28.