Mathematics 242 – Principles of Analysis
Final Exam Solutions – December 14, 2005

I.

A) The sequence (x_n) converges to L if for every $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ such that $|x_n - L| < \varepsilon$ whenever $n \geq N$.

B)

$$\lim_{n\to\infty} \frac{5n^2+3n+1}{n^2+4n} = \lim_{n\to\infty} \frac{5+\frac{3}{n}+\frac{1}{n^2}}{1+\frac{4}{n}} = 5$$

by the Algebraic Limit Theorem.

C) Given $\varepsilon > 0$, let $N > \frac{17}{\varepsilon}$. Then for all $n \geq N$, $\frac{17}{n} < \frac{17}{N} < \varepsilon$, so

$$\begin{aligned}\left|\frac{5n^2+3n+1}{n^2+4n} - 5\right| &= \frac{|-17n+1|}{n^2+4n}\\ &= \frac{17n-1}{n^2+4n}\\ &< \frac{17n}{n^2}\\ &= \frac{17}{n} < \varepsilon\end{aligned}$$

This shows the limit of the sequence is 5 as claimed.

II. First we show x_n is bounded above by 2 by induction. The base case is $n = 1$, where $x_1 = 1 \leq 2$ is clear. Then assume $x_k \leq 2$. We have $x_{k+1} = \sqrt{x_k+1} \leq \sqrt{2+1} = \sqrt{3} \leq 2$. So x_n is bounded above by 2 for all n.

Next we show that x_n is monotone increasing. We have $x_2 = \sqrt{2} > 1 = x_1$ for the base case. Next assume $x_k > x_{k-1}$. Then

$$x_{k+1} = \sqrt{x_k+1} > \sqrt{x_{k-1}+1} = x_k$$

So the sequence is monotone (strictly) increasing.

Finally, we know the sequence converges to some $L \leq 2$ by the monotone convergence theorem. To find L we take $\lim_{n\to\infty}$ in the equation $x_n = \sqrt{x_n+1}$, so $L = \sqrt{L+1}$, or $L^2 - L - 1 = 0$. This gives $L = \frac{1\pm\sqrt{5}}{2}$ by the quadratic formula. The limit must be positive, so $L = \frac{1+\sqrt{5}}{2}$.

III. The Bolzano-Weierstrass theorem is the statement that every bounded sequence has a convergent subsequence. See the class notes and text for the proof.

IV.

A) The infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums: $S_N = \sum_{n=1}^{N} a_n$ converges to some $L \in \mathbf{R}$.

B) This is a geometric series with first term $a = 15/7$ and ratio $r = 3/7$, which is less than 1 in absolute value. Hence the series converges to

$$\frac{a}{1-r} = \frac{15/7}{1-3/7} = \frac{15}{4}.$$

C) $f(n) = \frac{1}{n+\ln(n)}$ is decreasing with n and $\lim_{n\to\infty} \frac{1}{n+\ln(n)} = 0$. Hence the series converges by the Alternating Series Test. The series of absolute values is $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$. Since $\ln(n) < n$ for all $n \geq 1$, we have $n + \ln(n) < 2n$ and $\frac{1}{n+\ln(n)} > \frac{1}{2n}$. The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (it's a constant times the harmonic series). Hence by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ diverges too. With the alternating signs, the series is *conditionally convergent.*

D)

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2\cdot 2^n}{(-1)^n n^2 2^{n+1}}\right| \\
&= \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^2 \\
&= \frac{1}{2} < 1
\end{aligned}$$

Hence, the series converges absolutely.

V.

A) No, $f(x)$ is not continuous at $x = 0$. Let $x_n = \frac{1}{n}$ (all rational. Then $\lim_{n\to\infty} f(x_n) = \left(\frac{1}{n}\right)^2 - 1 = -1$. But if $y_n = \frac{\sqrt{2}}{n}$ (all irrational), then $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 2(\frac{\sqrt{2}}{n}) - 2 = -2$. Since the two sequential limits are different, $f(x)$ is not continuous at $x = 0$, since $\lim_{x\to 1} f(x)$ does not exist.

B) Yes, $f(x)$ is continuous at $x = 1$. Note that

$$|f(x) - f(1)| = \begin{cases} |x^2-1| = |x+1||x-1| & \text{if } x \text{ is rational} \\ 2|x-1| & \text{if } x \text{ is irrational} \end{cases}$$

Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/3\}$. Then if $|x-1| < \delta$, we have $0 < 1-\delta < x < 1+\delta < 2$, so $1 < x+1 < 3$. Hence in either case (x rational or irrational), $|f(x)-f(1)| < 3\cdot\varepsilon/3 < \varepsilon$ or $|f(x) - f(1)| < 2\varepsilon/3 < \varepsilon$. This shows that $\lim_{x\to 1} f(x) = f(1)$ so f is continuous at $x = 1$.

VI.

A) This is TRUE. By contraposition, if f changes sign on the interval, then the Intermediate Value Theorem shows $f(x) = 0$ for some $x \in [a, b]$. Hence if $f(x)$ is never zero, then it is either always positive or always negative.

B) This is TRUE. By the Extreme Value Theorem, f attains its minimum at some $c \in [a, b]$: $f(x) \geq f(c)$ for all $x \in [a, b]$. But we are given that $f(c) > 0$, so we can take $m = f(c)$.

C) This is FALSE. Counterexample: $f : (-1,1) \to \mathbf{R}$ is continuous on the open interval, but the range $f(-1,1) = [0,1)$ which is not open.

VII. By the definition,

$$
\begin{aligned}
f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\
&= \lim_{x\to 0} \frac{x^3 \sin(1/x) + 3x}{x} \\
&= \lim_{x\to 0} x^2 \sin(1/x) + 3 \\
&= 3
\end{aligned}
$$

(Note $\sin(1/x)$ is bounded so $\lim_{x\to 0} x^2 \sin(1/x) = 0$.)

VIII. For the regular partition P_n of $[1,3]$ we have $\Delta x = 2/n$ $x_i = 1 + 2i/n$ for $i = 0, \ldots, n$. The function $f(x) = x^2 + 5$ is increasing on the interval $[1,3]$, so

$$
U(f, P_n) = \sum_{i=1}^{n} \left((1 + 2i/n)^2 + 5 \right) \frac{2}{n}
$$

and

$$
L(f, P_n) = \sum_{i=0}^{n-1} \left((1 + 2i/n)^2 + 5 \right) \frac{2}{n}
$$

Hence $U(f, P_n) - L(f, P_n) = (14 - 6)\frac{2}{n} = \frac{16}{n}$. This can be made $< \varepsilon$ by taking $\frac{n > 18}{\varepsilon}$. Hence f is integrable on $[1,3]$. The value of the integral is

$$
\begin{aligned}
\int_1^3 x^2 + 5 \, dx &= \lim_{n\to\infty} U(f, P_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left((1 + 2i/n)^2 + 5 \right) \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(6 + 4i/n + 4i^2/n^2 \right) \frac{2}{n} \\
&= \lim_{n\to\infty} 12 + \frac{8}{n^2} \sum_{i=1}^{\infty} i + \frac{8}{n^3} \sum_{i=1}^{\infty} i^2 \\
&= \lim_{n\to\infty} 12 + \frac{4n(n+1)}{n^2} + \frac{4n(n+1)(2n+1)}{3n^3} \\
&= 12 + 4 + 8/3 = 56/3
\end{aligned}
$$

IX.

A) This limit computes $\frac{d}{dx}\ln(x)$ at $x = 1$. By calculus we know $\frac{d}{dx}\ln(x) = \frac{1}{x}$, so

$$\lim_{h\to 0} = \frac{\ln(1+h)}{h} = 1.$$

B) Writing $h = 1/n$, this shows

$$1 = \lim_{n\to\infty} n \cdot \ln\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n$$

Write $x_n = \ln\left(1 + \frac{1}{n}\right)^n$. Then the sequential criterion for continuity shows

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} e^{x_n} = e^{\lim_{n\to\infty} x_n} = e.$$

Extra Credit. Since f is continous, it is integrable on $[1, B]$ for all $B > 1$. Since f is decreasing, the lower sum for the partition $P = \{1, 2, \ldots, N\}$ on $[1, N]$ is the sum

$$\sum_{n=2}^{N} f(n) = L(f, P) < \int_1^N f(x)\ dx < \lim_{B\to\infty}\int_1^B f(x)\ dx = L$$

Hence the partial sums of the series are bounded above by a fixed number L. By the Monotone Convergence Theorem, the partial sums converge to a finite limit, so $\sum_{n=1}^{\infty} f(n)$ converges.

To see that $\sum_{n=1} \frac{1}{n^\alpha}$ converges for all $\alpha > 1$, note that

$$\lim_{B\to\infty}\int_1^B \frac{1}{x^\alpha}\ dx = \lim_{B\to\infty} \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_1^B = \lim_{B\to\infty} \frac{B^{1-\alpha} - 1}{1-\alpha}$$

Since $\alpha > 1$, $1 - \alpha < 0$ so this limit exists, and equals $\frac{1}{\alpha-1}$. Hence the sum converges by the first part of the question.