Mathematics 242 – Principles of Analysis
Exam 3 Solutions – December 2, 2005

I. $f(x) = x^2 - 6x + 3$.

A) Let $\varepsilon > 0$ and $\delta < \min\{1, \varepsilon/5\}$. If $|x-1| < \delta < 1$, then $0 < x < 2$, so $|x-5| < 5$. Then since we also have $|x-1| < \varepsilon/5$:

$$|f(x) - (-2)| = |x^2 - 6x + 5| = |x-1||x-5| < (\varepsilon/5) \cdot 5 = \varepsilon.$$

This shows $\lim_{x\to 1} f(x) = -2$.

B) Note that $f'(x) = 2x - 6 \geq 0$ on the interval $[3,4]$. Therefore f is monotone increasing. By the result from question C on Discussion 3, f is integrable on that interval. (Alternate way: f is continuous on $\mathbf{R}$, so by the theorem we proved in class on 11/30, f is integrable on every finite interval.)

II.

A) IVT: Let f be continuous on $[a,b]$. If L is any real number (strictly) between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ such that $f(c) = L$.

B) $f(x)$ is continuous because $f'(x) = -a_2 \sin(x)$ exists for all real x. Moreover $f(0) = a_1 + a_2 > 0$ while $f(\pi) = a_1 - a_2 < 0$. Taking $L = 0$ in the IVT, there exists $c \in (0, \pi)$ such that $f(c) = 0$.

III. Since $s = \sup f([a,b])$, if $\varepsilon > 0$, then $s - \varepsilon$ is not an upper bound for $f([a,b])$. So there exists $x \in [a,b]$ such that $s - \varepsilon < f(x) \leq s$. Apply this with $\varepsilon = \frac{1}{n}$ for each $n \in \mathbf{N}$. This gives a sequence (x_n) in $[a,b]$ such that $s - \frac{1}{n} < f(x_n) < s$ for all $n \in \mathbf{N}$. Since $[a,b]$ is a bounded interval, the Bolzano-Weierstrass theorem implies that there is a subsequence x_{n_k} of this sequence converging to some c. By the above $\lim_{k\to\infty} f(x_{n_k}) = s$. Since the interval is closed, the limit c is also in the interval $[a,b]$. But then by the sequential version of continuity

$$s = \lim_{k\to\infty} f(x_{n_k}) = f(c).$$

IV.

A) This is FALSE. Let x_n be any sequence of rational numbers converging to 3. Then $\lim_{n\to\infty} f(x) = 9$. On the other hand, if y_n is a sequence of irrational numbers converging to 3, then $\lim_{n\to\infty} f(y_n) = 0$. Since these two sequential limits are different the limit of the function does not exist.

B) This is TRUE. We have

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} \frac{x^2}{x} = x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Either way, if $|x| < \delta = \varepsilon$, then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| < \varepsilon$$

whenever $|x| < \varepsilon$. So the limit of the difference quotient exists and

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

C) This is FALSE. If we pick any partition $P = \{x_i : i = 0, \dots, n\}$ of the interval, then each subinterval contains rational numbers arbitrarily close to the right-hand endpoint of the subinterval. Hence $M_i = x_i^2$. On the other hand $m_i = 0$ since every subinterval also contains irrational numbers. Since all the $x_i^2 > 1$, The difference

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} x_i^2 \Delta x_i > \sum_{i=1}^{n} \Delta x_i = 1.$$

This shows f is not integrable on $[1, 2]$.

D) This is FALSE. $s = \sup f([-\sqrt{2}, \sqrt{2}]) = 2$ but there is no x in the interval where $f(x) = 2$. (Note $\pm\sqrt{2}$ are irrational, so $f(\pm\sqrt{2}) = 0$).