Mathematics 242 – Principles of Analysis
Discussion 1 – Set Operations
September 8, 2004

Background

The language of (modern) mathematics is the language of *sets*. Today, we want to review some of the basic ideas about sets and the most common operations on sets. Recall that a set is essentially just a *collection of objects*. To specify a set, we can either list the elements, e.g. $A = \{1, 2, 3, 4\}$ for the set of natural numbers less than 5, or we can give a "test" for determining which objects are in the set. The usual notation for this is called "set-builder" notation. Here is an example:

$$A = \{x : x \text{ is a natural number and } x < 5\},$$

or

$$A = \{x : x \in \mathbf{N} \text{ and } x < 5\},$$

($\mathbf{N}$ is a common name used for the set of natural numbers. Similarly, we often write $\mathbf{Z}$ for the set of integers, $\mathbf{Q}$ for the set of all rational numbers, $\mathbf{R}$ for the set of all real numbers.) Either of these equations is read as "A is the set of all x such that x is a natural number and $x < 5$."

The proposition that an object x is an element of the set A is written symbolically as $x \in A$, and its negation is written as $x \notin A$. We emphasize that $x \in A$ should be a *proposition* – unambiguously true or false, and not both. For instance "I am an element of the set of tall people" does not qualify (at least not without a more precise definition of what the set of tall people is (what the exact criterion for tallness is). The set with no elements (i.e. the set where $x \in A$ is always false) is called the *empty set*, written $\emptyset$.

Some additional notation: Given sets A, B, we can form

- the *union*
$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$
- the *intersection*
$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$
- the *set difference*
$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Note the logical connectives used here.

For example, let $A = \{1, 2, 5, 6\}, B = \{2, 5, 7\}, C = \{2, 3, 4\}$ (all subsets of the set of natural numbers). Then

$$A \cup B = \{1, 2, 5, 6, 7\}$$
$$B \cap C = \{2\}$$
$$A \setminus C = \{1, 5, 6\}$$

I. Using set-builder notation, how would you write:
A) The set of all real numbers between 1 and 3 (inclusive)?
B) The set of all natural numbers that satisfy $x^2 - 5x + 6 = 0$.

II. Let $A = \{2, 3, 6\}$, $B = \{x : x \text{ is an even natural number and } 3 < x < 7\}$, $C = \{1, 3, 6, 8\}$. Find
A) $A \cup B$.
B) $B \setminus (A \cap C)$
C) $(B \setminus A) \cap C$.

III. Using the same A, B, C from question II, decide whether the following statements are true or false, and give a reason for your conclusion in each case. Notation: $X \subseteq Y$ is the proposition "the set X is contained in the set Y" – that is, every element in X is also an element of Y.

A) $(A \cap B) \subseteq C$
B) $\emptyset \in A$ (What is the difference between saying $\emptyset \subseteq A$ and $\emptyset \in A$?)
C) $(B \setminus A) \subseteq C$
D) $A \cap B \cap C = 6$. (Careful, what "type of object" is 6? Is it the same type of object as the left side of the equation?)

IV. The set operations above satisfy a number of different properties that will be useful to us. See Theorem 5.13 in the text for a few. To practice some of the ideas about proofs we discussed for the past few days, let's consider 5.13 g: For all sets A, B, C,

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \tag{1}$$

(one of "DeMorgan's Laws"). To prove an equality of sets $X = Y$, one basic approach is to show both inclusions $X \subseteq Y$ and $Y \subseteq X$. This will give the basic outline of the proof, one paragraph to show each inclusion. Within each paragraph we will use another standard idea about showing inclusions:
A) Explain why the statement $X \subseteq Y$ is equivalent to the statement $x \in X \Rightarrow x \in Y$.
B) "Fill in the blanks" to give a complete proof of (1):

Proof: First we show $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$. Let $x \in A \setminus (B \cap C)$. We want to show that $x \in (A \setminus B) \cup (A \setminus C)$ follows from this. By the definition of set difference, $x \in A \setminus (B \cap C)$ means $x \in$ __ and $x \notin$ _________. By 1.6 a on page 6 of the text, $x \notin B \cap C$ is equivalent to $x \notin B$ __ $x \notin C$. But then $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. By definition, again, this means $x \in A \setminus B$, or $x \in A \setminus C$. Hence by the definition of the union, ___________.

Next, we show $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$. Let ________. Then $x \in A$ and $x \notin B$, or $x \in A$ and _______. By the definition of the set difference, this says $x \in A$ and either $x \notin B$, or $x \notin C$ (or both). By 1.6 a again, we get $x \in A \setminus (B \cap C)$.

C) Develop your own proof that if A, B are any sets and $A \cap B = A$, then $A \subset B$.

Assignment

Group write-ups due Monday, September 13.