Mathematics 242 – Principles of Analysis
Practice Final Examination

I. Let $A = \{x^2 - 2 : -1 < x < 3\}$ and $B = \{x : |x-1| < 4\}$. Find $\sup(A \cup B)$ and $\inf(A \cap B)$.

II.

A) State the ϵ, N definition of convergence for a sequence of real numbers.
B) Identify $\lim_{n\to\infty} \frac{5n+1}{n+4}$.
C) Show that your result in part B is correct using the definition.

III.

A) Show that if x_n is a monotone increasing sequence of real numbers that is bounded above, then x_n converges to some real number.
B) What general theorem could you use to deduce that the sequence $x_n = (-1)^n + \frac{(-1)^{n+1}}{n^2}$ has convergent subsequences?
C) Let $x_n = (-1)^n + \frac{(-1)^{n+1}}{n^2}$. Find a monotone increasing subsequence of (x_n) and identify the limit.

IV.
A) Show that if $f(x)$ is continuous at $x = c$, and x_n is any sequence satisfying $\lim_{n\to\infty} x_n = c$, then $\lim_{n\to\infty} f(x_n) \to f(c)$.
B) Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is a rational number} \\ -x & \text{if } x \text{ is an irrational number} \end{cases}$$

Is f continuous at $x = 1$? Why or why not? What about at $x = 0$?

V.
A) State and prove the Intermediate Value Theorem.
B) Show that for every real y_0, there exists a solution x of the equation $\frac{e^x - e^{-x}}{2} = y_0$

VI. (15) Using the definition, compute $f'(x)$ for $f(x) = \frac{1}{(x+3)^2}$.

VII.
A) (20) In this part you may use the summation formulas:

$$\sum_{i=1}^{n} 1 = n \quad \sum_{i=1}^{n} i = \frac{n^2+n}{2} \quad \sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}.$$

Show that $f(x) = x^2 + 5$ is integrable on $[a,b] = [0,1]$ by considering upper and lower sums for f and determine the value of $\int_0^1 x^2 + 5\, dx$.
B) (15) State and prove the Fundamental Theorem of Calculus, “part 1.”

VIII.

A) State the definition of convergence for an infinite series $\sum_{n=1}^{\infty} a_n$.

B) Suppose that in an infinite series $\sum_{n=1}^{\infty} a_n$, $a_n < 0$ for all n. Show that if the partial sum s_N satisfies $s_N > B$ for all N, then $\sum_{n=1}^{\infty} a_n$ converges. (Hint: What kind of sequence is $\{s_N\}$?)

C) Does the series $\sum_{n=1}^{\infty} \frac{5 \cdot 3^n}{7^n}$ converge? If so, what is the sum of the series?

D) Use the Integral Test to determine whether

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges.

E) Use the Comparison Test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^{3.9}+1}$$

converges absolutely.

IX.

A) Noting that

$$\lim_{h \to 0} \frac{\ln(1+h)}{h} = \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h},$$

evaluate the limit. (You may use any calculus facts you need here without proof.)

B) Let $h = \frac{1}{n}$ in the limit from part A, and use that result and the theorem from question IV A on this exam with $f(x) = e^x$ to compute

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$