Mathematics 242 – Principles of Analysis
Discussion 2 – Solutions
October 7, 2004

Discussion Questions

A) The second defining property of the supremum $s = \sup(S)$ above says: if $s' < s$, then there exists $x \in S$ such that $x > s'$. Show that this statement is equivalent to saying: If t is any upper bound for S, then $t \geq s$. (Note and Hint: In fact, many analysis books *define* the supremum using this alternate form, because it clearly shows that $\sup(S)$ is the "least upper bound" for S. The relation between these statements comes from one of our basic logical equivalences!)

Solution: The form given in question A for saying $s = \sup(S)$ is the *contrapositive* of the book's definition: Book: If $s' < s$, then s' is not an upper bound for S (that is, there exists an $x \in S$ such that $x > s'$). Contrapositive of this is: If s' is an upper bound for S, then $s' \geq s$. Replacing s' by t we get exactly what is here.

B) Let S be a nonempty bounded subset of $\mathbf{R}$. Show that $s = \sup(S)$ is *unique*. Method: Assume that two real numbers s, s' both satisfy the definition, then deduce that s must equal s'.

Solution: Using the form from problem A, let s and s' both satisfy the defining properties of $\sup(S)$. Since s is $\sup(S)$, and s' is an upper bound for S, we get from A, $s \leq s'$. Similarly, since s' is $\sup(S)$ and s is an upper bound for S, $s' \leq s$. These two inequalities show $s = s'$.

C) Prove that if $x < y$ are any real numbers, then there exist *infinitely many* rational numbers r with $x < r < y$. Hint: Last time we showed that there is one such rational number. Try to extend that proof.

Solution: In class we showed that given any reals $x < y$, there exists a rational number r_1 with $x < r_1 < y$. Repeat the same argument, with r_1 playing the role of y. There exists a rational number r_2 with $x < r_2 < r_1 < y$. Continuing in the same way, we can find a sequence of rational numbers r_n, $n \in \mathbf{N}$ such that $x < \cdots < r_n < \cdots < r_2 < r_1 < y$ for all n. The r_i are distinct by construction, so there are infinitely many rationals between x and y (a whole denumerable set of them).

D) Let a/b be a rational number written as a fraction in lowest terms with $0 < a/b < 1$.

1) Show using the Archimedean Property that there exists an integer n such that

$$\frac{1}{n+1} \leq \frac{a}{b} \leq \frac{1}{n}.$$

Solution: Consider the number $b/a > 1$. By the Archimedean Property of $\mathbf{N}$, there exists a natural number n such that $n \leq b/a \leq n+1$. Then inverting (and correspondingly reversing the inequalities), we have

$$\frac{1}{n+1} \leq \frac{a}{b} \leq \frac{1}{n}$$

as claimed. Note: We cannot have equality in both cases, so we could always write one of these as a strict inequality. If a/b happens to equal the reciprocal of a natural number we could make $a = 1/(n+1)$, for instance.

2) If n is chosen as in part 1, show that $a/b - 1/(n+1)$ is a fraction that when written in lowest terms has numerator is less than a.

Solution: We have $a/b - 1/(n+1) = (a(n+1) - b)/(b(n+1))$. If this fraction is not already in lowest terms, then canceling common factors can only make the numerator *smaller*. Hence to prove what we want here, it suffices to show that

$$a(n+1) - b < a.$$

From the proof of part 1, $an < b \leq a(n+1)$, so subtracting $a(n+1)$ everywhere, $-a < b - a(n+1) < 0$. Reversing signs (and switching the inequalities correspondingly), $a > a(n+1) - b$, which is what we wanted to show.

3) Use the Principle of Strong Induction from Problem Set 3 to show that every rational number a/b as above can be written as a sum:

$$\frac{a}{b} = \frac{1}{n_1} + \cdots + \frac{1}{n_k}$$

for some distinct natural numbers n_i. (For instance $49/90 = 1/3 + 1/9 + 1/10$.)

Solution: The result from part 2 says that if we find the integer n with $1/(n+1) \leq a/b < 1/n$, then the difference $a/b - 1/(n+1)$, written in lowest terms, must have a smaller numerator than a/b. This means that we should try to set up an induction argument where the "induction variable", so to speak, is the integer a in the numerator of the fraction.

The base case for the induction is numbers of the form $a/b = 1/b$ (where $a = 1$). These are already written in unit fraction, or "Egyptian" form. So the base case requires no proof (:)). Now consider any $0 < a/b < 1$ in lowest terms. By part 2, there exists an integer $n+1$ such that $a/b - 1/(n+1)$ has numerator $< a$ when written in lowest terms. By strong induction, this means that we can write

$$\frac{a}{b} - \frac{1}{(n+1)} = \frac{1}{n_1} + \cdots + \frac{1}{n_k}$$

for some distinct integers $n_1, \ldots, n_k$. But then

$$\frac{a}{b} = \frac{1}{(n+1)} + \frac{1}{n_1} + \cdots + \frac{1}{n_k}$$

and the only thing left to prove is that $n+1, n_1, \ldots, n_k$ must all be *distinct*. Since $n_1, \ldots, n_k$ are distinct, this just means that we need to show $n+1 \neq n_i$ for $i = 1, \ldots, k$. Aiming for a contradiction, suppose on the contrary that $n+1 = n_i$ for some i, and for convenience, renumber so that $i = 1$. Then $a/b - 2/(n+1) = 1/n_2 + \cdots + 1/n_k > 0$. But for $n > 1$, we always have $2/(n+1) > 1/n$. This implies that $a/b - 1/n > a/b - 2/(n+1) > 0$. But note that n was chosen to make $1/(n+1) \leq a/b < 1/n$, so $a/b - 1/n < 0$. Hence, we have found a contradiction because $a/b - 1/n$ cannot be both positive and negative. Hence the denominators $n+1, n_1, n_2, \ldots, n_k$ are all distinct.