Mathematics 242 – Principles of Analysis
Discussion 2 – $\sup(S)$ continued
September 29, 2004

Background

Last time, we introduced the *supremum* or *least upper bound* of a subset $S \subseteq \mathbf{R}$. Recall, we say that $s = \sup(S)$ if

- s is an upper bound of S – $x \leq s$ is true for all $x \in S$, and
- if $s' < s$, then there exists $x \in S$ such that $x > s'$.

The *completeness axiom*, the crucial property of the real numbers, is:

Completeness Axiom. *If S is nonempty and bounded above in $\mathbf{R}$, then $\sup(S)$ exists in $\mathbf{R}$.*

That is, *every nonempty set that is bounded above in $\mathbf{R}$ has a least upper bound in $\mathbf{R}$.* Today, we want to study further properties of the supremum.

Discussion Questions

A) The second defining property of the supremum $s = \sup(S)$ above says: if $s' < s$, then there exists $x \in S$ such that $x > s'$. Show that this statement is equivalent to saying: If t is any upper bound for S, then $t \geq s$. (Note and Hint: In fact, many analysis books *define* the supremum using this alternate form, because it clearly shows that $\sup(S)$ is the "least upper bound" for S. The relation between these statements comes from one of our basic logical equivalences!)

B) Let S be a nonempty bounded subset of $\mathbf{R}$. Show that $s = \sup(S)$ is *unique*. Method: Assume that two real numbers s, s' both satisfy the definition, then deduce that s must equal s'.

C) Prove that if $x < y$ are any real numbers, then there exist *infinitely many* rational numbers r with $x < r < y$. Hint: Last time we showed that there is one such rational number. Try to extend that proof.

D) Let a/b be a rational number written as a fraction in lowest terms with $0 < a/b < 1$.

   1) Show using the Archimedean Property that there exists an integer n such that

$$\frac{1}{n+1} \leq \frac{a}{b} \leq \frac{1}{n}.$$

   2) If n is chosen as in part 1, show that $a/b - 1/(n+1)$ is a fraction that when written in lowest terms has numerator is less than a.

3) Use the Principle of Strong Induction from Problem Set 3 to show that every rational number a/b as above can be written as a sum:

$$\frac{a}{b} = \frac{1}{n_1} + \cdots + \frac{1}{n_k}$$

for some distinct natural numbers n_i. (For instance $49/90 = 1/3 + 1/9 + 1/10$.)

Such fractions are often called "Egyptian fractions" because the ancient Egyptians did all of their rational number arithmetic using them (for some reason they didn't like fractions with a numerator different from 1). They wrote all fractions this way, then computed sums and products via tabulations that have survived in ancient papyrus manuscripts – the "Rhind papyrus" is probably the most famous example. It is one of the oldest surviving ancient mathematical documents. There is an interesting discussion of this online at

`http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Egyptian_papyri.html`

Assignment

Group writeups due by 5:00pm on Monday, October 4.