Mathematics 242 – Principles of Analysis
Problem Set 2 – Selected Solutions
September 27, 2004

8.4e/ We must construct a bijective mapping $f : (0, 1) \to \mathbf{R}$. In order to get the whole real line as the image of a function on a finite interval, we expect the graph must have vertical asymptotes at both $x = 0$ and $x = 1$, say with $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to 1^-} f(x) = +\infty$. We recall from calculus that the tangent function has a graph with the right general shape. However $\tan(x)$ has successive vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$. So we must shift and "squeeze" the graph horizontally to place the asymptotes at $x = 0$ and $x = 1$. The function $f(x) = \tan(\pi x - \pi/2)$ will work for this purpose. Note that $x = 0$ corresponds to $-\pi/2$ and $x = 1$ corresponds to $-\pi/2$. Since $\tan(x)$ is strictly increasing on the open interval $(-\pi/2, \pi/2)$, it is injective. It is continuous and surjective onto $\mathbf{R}$. (A full proof of this requires a result called the Intermediate Value Theorem that we will prove in a few weeks.) Hence our f is a bijection from $(0, 1)$ to $\mathbf{R}$.

8.11/ a) We must show that there is a bijection $f : (0, 1) \to (m, n)$. The linear function with $f(0) = m$ and $f(1) = n$ will work for this purpose, since both endpoints of both intervals are omitted. We obtain $f(x) = (n - m)x + m$. This is injective since

$$f(x) = f(x') \Rightarrow (n - m)x + m = (n - m)x' + m \Rightarrow (n - m)x = (n - m)x' \Rightarrow x = x'.$$

The surjectivity can be proved directly here (without recourse to the Intermediate Value Theorem). Given any $y \in (m, n)$, we can solve the equation $y = (n - m)x + m$ for x: $x = \frac{y-m}{n-m}$. If $m < y < n$, we see $0 < x < 1$.

b) Using part a, we have bijections $f : (0, 1) \to (m, n)$ and $g : (0, 1) \to (m', n')$. The function $h = g \circ f^{-1}$ maps (m, n) to (m', n'). Moreover h is a bijection since f^{-1} and g are bijections.

10.15/ We assume that $p(n)$ is a collection of statements, one for each $n \in \mathbf{N}$, satisfying

1) $p(1)$ is true, and
2) $p(1) \wedge p(2) \wedge \cdots \wedge p(k) \Rightarrow p(k + 1)$ for all $k \geq 1$.

We claim that $p(n)$ is true for all $n \in \mathbf{N}$ (the principle of strong induction). The proof is by contradiction. Let $F = \{n : p(n) \text{ is false}\}$, so we assume F is nonempty. By the Well-Ordering Principle, F has a smallest element n_0. By assumption 1, $n_0 > 1$. Hence $n_0 - 1 \in \mathbf{N}$, and $1, 2, \ldots, n_0 - 1 \in \mathbf{N} \setminus F$, so $p(1), \ldots, p(n_0 - 1)$ are all true. Then assumption 2 shows that $p(1) \wedge \cdots \wedge p(n_0 - 1)$ implies $p(n_0)$ is also true. But this is a contradiction, since $n_0 \in F$. Therefore F must be empty, and $p(n)$ is true for all $n \in \mathbf{N}$.

10.18 a) Finding a common denominator, we have

$$\begin{aligned}\binom{n}{r} + \binom{n}{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\ &= \frac{n!((n-r+1)+r)}{r!(n-r+1)!} \\ &= \frac{n!(n+1)}{r!(n-r+1)!} \\ &= \frac{(n+1)!}{r!(n-r+1)!} \\ &= \binom{n+1}{r}\end{aligned}$$

b) We must prove that

$$(a+b)^n = \sum_{i=0} \binom{n}{i} a^{n-i}b^i = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n$$

This proof will be by induction on n. The base case $n = 1$ is true since

$$(a+b)^1 = a + b = \binom{1}{0}a + \binom{1}{1}b$$

Now we assume as our induction hypothesis that

$$(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{k}b^k.$$

Now we consider $(a+b)^{k+1}$ and use the induction hypothesis:

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k \\ &= (a+b)\left(\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{k}b^k\right) \\ &= \binom{k}{0}a^{k+1} + \binom{k}{1}a^k b + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k}ab^k \\ &\qquad \binom{k}{0}a^k b + \binom{k}{1}a^{k-1}b^2 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1}\end{aligned}$$

We collect like terms and use part a. Then since $\binom{k}{0} = \binom{k+1}{0} = \binom{k}{k} = \binom{k+1}{k+1} = 1$,

$$= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k+1}b^{k+1}$$