Mathematics 242 – Principles of Analysis
Selected Solutions – Problem Set 2
September 20, 2004

Section 6/8. Let R be the relation on $\mathbf{R}^2$, defined by

$$(a, b)R(c, d) \Leftrightarrow a + d = b + c$$

First, we show that R is an equivalence relation. First, it is reflexive since $a + b = b + a$ implies $(a, b)R(a, b)$ for all (a, b). Next, it is symmetric, since if $(a, b)R(c, d)$ then $a + d = b + c$. Hence $c + b = d + a$ so $(c, d)R(a, b)$. Finally, it is transitive since if $(a, b)R(c, d)$ and $(c, d)R(e, f)$ then $a + d = b + c$ and $c + f = d + e$. These equations imply that $a + d + f = b + c + f = b + d + e$, so $a + f = b + e$, which shows $(a, b)R(e, f)$. The equivalence class of $(7, 3)$ is the set of all (x, y) such that $(x, y)R(3, 7)$, or $x + 3 = y + 7$. This is the line defined by the equation $y = x - 7$ in the plane.

Section 7/8. Note that the problem asks for functions $f : \mathbf{N} \to \mathbf{N}$. This means that the domains and ranges must be subsets of the set of natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$.

a) The function

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ n - 1 & \text{if } n \geq 2 \end{cases}$$

is not injective since $f(1) = f(2) = 1$. But it is surjective, since every natural number is in its range.

b) An injective, but not surjective function is $f(n) = 2n$. The range is the set of even natural numbers, so f is not surjective. But f is injective since $f(n_1) = f(n_2)$ implies $2n_1 = 2n_2$, so $n_1 = n_2$.

c) A function that is neither injective nor surjective is

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$$

The range is the set of even natural numbers, so f is not surjective. Moreover, f is not injective, since $f(2) = f(1) = 2$.

d) $f(n) = n$ is bijective.

Section 7/10.

a) We want an example where $f : A \to B$ and $f^{-1}(f(C)) \neq C$ for some $C \subseteq A$. Let $f(x) = x^2$, and let $C = \{1\}$. We have $f^{-1}(f(C)) = \{x : x^2 = 1\} = \{-1, 1\}$.

b) We want an example where $f(f^{-1}(D)) \neq D$ for some subset $D \subset B$. Let $f(x) = x^2$ again, and take $D = \{-1\}$. Then $f^{-1}(D) = \{x : x^2 = -1\} = \emptyset$ (as long as we take the domain of f to be the set of real numbers, or a subset of the reals). Hence $f(f^{-1}(D)) = \emptyset \neq D$.

c) We want an example where $f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$ for subsets $C_1, C_2 \subset A$. Take $f(x) = x^2$ again, and let $C_1 = \{1\}$, $C_2 = \{-1\}$. Then $f(C_1 \cap C_2) = \emptyset$, but $f(C_1) = f(C_2) = \{1\}$, so $f(C_1) \cap f(C_2) = \{1\}$.

Section 7/12. Let $f : A \to B$.

a) We want to show that f injective implies $f^{-1}(f(C))$ for all subsets $C \subset A$. We know $C \subseteq f^{-1}(f(C))$ for all mappings. Hence, we must show that f injective implies $f^{-1}(f(C)) \subseteq C$. Let $x \in f^{-1}(f(C))$. By definition, this means $f(x) \in f(C)$, so $f(x) = f(x')$ for some $x' \in C$. Since f is injective, this implies $x = x'$, so $x \in C$. We have shown $f^{-1}(f(C)) \subseteq C$.

b) Similarly, we want to show that f surjective implies that $f(f^{-1}(D)) = D$ for all $D \subseteq B$. We know that $f(f^{-1}(D)) \subset D$ for all mappings f and all D. So, we need to show that f surjective implies that $D \subseteq f(f^{-1}(D))$. So let $y \in D$. Since $D \subset B$ and f is surjective, this means that $y = f(x)$ for some $x \in A$. But then by definition, this implies that $x \in f^{-1}(D)$, and $y = f(x) \in f(f^{-1}(D))$. This shows that $D \subseteq f(f^{-1}(D))$.

Section 7/13. We assume that f, g are injective. Let $g(f(x)) = g(f(x'))$. Since g is injective, $f(x) = f(x')$. But then, since f is injective, $x = x'$. This shows that the composition $g \circ f$ is injective also.

Section 7/20.

a) If $f : A \to B$ has a left inverse function g and $f(x) = f(x')$, then $x = g(f(x)) = g(f(x')) = x'$. This implies that f is injective. Conversely, suppose f is injective. For each $y \in B$, y is either equal to $f(x)$ for some $x \in A$ or else $y \neq f(x)$ for all $x \in A$. In the first case, since f is injective, there is only one such x, so we can define $g(y) = x$, where x is the unique element of A for which $f(x) = y$. If $y \neq f(x)$ for all $x \in A$, then pick some $a \in A$ arbitrarily and define $g(y) = a$. Doing this we get a function $g : B \to A$, and $g(f(x)) = x$ for all $x \in A$. Hence g is a left inverse of f.

b) If $f : A \to B$ has a right inverse function g, then $f(g(y)) = y$ for all $y \in B$. This says that every $y \in B$ is in the range of f, so f is surjective. Conversely, if f is surjective, then we must construct a right inverse function g. To do this, since f is surjective, for every $y \in B$, there exists (at least one) $x \in A$ such that $f(x) = y$. For each y, pick any one such x and define $g(y) = x$. This gives a function $g : B \to A$ with $f(g(y)) = y$ for all $y \in B$. Hence g is a right inverse for f.