Mathematics 242 – Principles of Analysis
Solutions – Final Examination
December 16, 2004

I. We have

$$A = \{\cos(x) : x \in [0, 3\pi/4]\} = [-\sqrt{2}/2, 1] \doteq [-.707, 1]$$

and

$$B = \{x : 1 < x^2 < 4\} = (-2, -1) \cup (1, 2)$$

so:

A) $A \cup B = (-2, -1) \cup [-\sqrt{2}/2, 2)$.

B) For each $x \in A$, $|x - 2|$ represents the distance along the number line from x to 2. So

$$C = \{|x - 2| : x \in A\} = [1, 2 + \sqrt{2}/2]$$

The least upper bound of C is $2 + \frac{\sqrt{2}}{2} \doteq 2.707$.

II. A) We say $\lim_{n\to\infty} x_n = L$ if for all $\varepsilon > 0$, there exist N such that $|x_n - L| < \varepsilon$ for all $n \geq N$. (The N can be taken to be a natural number or an arbitrary real, as convenient.)

B) We have

$$\lim_{n\to\infty} \frac{3n^2 + n}{n^2 + 1} = \lim_{n\to\infty} \frac{3n^2 + n}{n^2 + 1} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{3 + 1/n}{1 + 1/n^2} = 3.$$

To prove this, note that

$$\left|\frac{3n^2 + n}{n^2 + 1} - 3\right| = \frac{|n - 3|}{n^2 + 1}$$

There are several ways to proceed from here. For instance, using the triangle inequality in the numerator, we could say:

$$\frac{|n - 3|}{n^2 + 1} \leq \frac{n + 3}{n^2 + 1},$$

which is

$$\leq \frac{2n}{n^2 + 1} < \frac{2n}{n^2} = \frac{2}{n}$$

as soon as $n \geq 3$. Here's the finished proof based on this idea: Let $\varepsilon > 0$, and let $N > \max(3, \frac{2}{\varepsilon})$ Then for all $n \geq N$, we have

$$\left|\frac{3n^2 + n}{n^2 + 1} - 3\right| = \frac{|n - 3|}{n^2 + 1} \leq \frac{n + 3}{n^2 + 1} < \frac{2}{n} < \varepsilon.$$

III. A) The sequence $x_n = \sin(n)$ is bounded since $|\sin(n)| \leq 1$ for all n. Hence the Bolzano-Weierstrass Theorem says that there exists a convergent subsequence $x_{n_k} = \sin(n_k)$ for some sequence of integers n_k.

B) Consider the sequence $\cos(n_k)$, using the n_k from part A. This is also a bounded sequence since $|\cos(n_k)| \le 1$ for all k. Hence the Bolzano-Weierstrass Theorem implies that there is a convergent subsequence of this sequence, say $y_{n_{k_\ell}} = \cos(n_{k_\ell})$. Note that this sequence comes from a subsequence of the index sequence n_k. Hence $x_{n_{k_\ell}}$ is a subsequence of the convergent sequence from part A. So since any subsequence of a convergent sequence is convergent too, the $x_{n_{k_\ell}}$ is also convergent.

Note: A lot of people lost a lot of points on this one because they did not take into account that the n inside the sin and cos in the definitions of the sequences x_n and y_n is the same as the index for the sequences. You have to consider plugging *integer values* into the two trig functions. Things like $n\pi/2$ are NOT INTEGERS!

IV. A) Let $f : D \to \mathbf{R}$ and let c be an accumulation point of D. Then we say $\lim_{x\to c} f(x) = L$ if for all $\varepsilon > 0$, there exist $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in D$ with $0 < |x - c| < \delta$.

B) The limit here is 0. Proof: Let $\varepsilon > 0$ and let $\delta = \sqrt{\varepsilon}$. Then for all x with $0 < |x - 0| < \sqrt{\varepsilon}$, we have (using $|\cos(1/x)| \le 1$ for all $x \ne 0$)

$$|x^2 \cos(1/x) - 0| = |x|^2 |\cos(1/x)| \le |x^2| < (\sqrt{\varepsilon})^2 = \varepsilon.$$

Note: It is not correct to evaluate the limit using the limit product rule here. The problem is that $\lim_{x\to 0} \cos(1/x)$ *does not exist.* However, $\cos(1/x)$ is bounded and the other factor x^2 goes to zero, so the product goes to zero.

V. A) $f(0) = 0$ and $f(2) = 1$ by computing.

B) The denominator $x^4 + 48$ is nonzero for all $x \in \mathbf{R}$. Hence $f(x) = \frac{32x}{x^4+48}$ is continuous at all x. By the IVT on the interval $[0, 2]$, for each k with $0 < k < 1$, there is at least one $x \in (0, 2)$ such that $f(x) = k$. To find a second x satisfying this condition, note that $\lim_{x\to+\infty} f(x) = 0$. Hence by the IVT again, there is also at least one additional solution of $f(x) = k$ for $x \in (2, +\infty)$.

C) (*Note:* We are interested in applying this statement to the $f(x)$ from the problem, which is differentiable at all x. So it suffices to consider the contrapositive statement, *assuming that f is differentiable*: If for some k, $f(x_1) = f(x_2) = k$ for some $x_1 \ne x_2$, then $f'(x) = 0$ for some x between x_1 and x_2. This is exactly the situation of the special case of the MVT known as Rolle's Theorem. So because the contrapositive statement is equivalent to this one, we have: If f is differentiable and $f'(x) \ne 0$ for all $x \in I$, then for all k, the equation $f(x) = k$ has at most one solution $x \in I$.

D) Our function f has derivative

$$f'(x) = \frac{1536 - 96x^2}{x^4 + 48}$$

(quotient rule for derivatives!). This is zero for $x > 0$ only at $x = 2$. Hence by part C, on the intervals $(0, 2)$ and $(2, +\infty)$, there can only be one solution of $f(x) = k$ in each interval. This means that there are exactly two of them all together.

VI. The most economical way to show that $f(x) = x^2 + x - 1$ is integrable on $[0, 3]$ is to note that $f(x)$ is monotone increasing on that interval ($f'(x) = 2x + 1 > 0$ for all $x \in [0, 3]$.) Hence as in the proof of our general theorem, for the regular partition P_n (with $\Delta x = 3/n$),

$$|U(f, P_n) - L(f, P_n)| = |f(3) - f(0)|\frac{3}{n} = \frac{36}{n}.$$

Since this goes to zero as $n \to \infty$, we know f is integrable. The value is computed using the upper sum, for instance. As above, $\Delta x = 3/n$, and the points in the partition are $P_n = \{3i/n : i = 0, 1, \ldots, n\}$. Then

$$\begin{aligned}
U(f, P_n) &= \sum_{i=1}^{n} \left(\left(\frac{3i}{i} \right)^2 + \frac{3i}{i} - 1 \right) \frac{3}{n} \\
&= \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \frac{9}{n^2} \sum_{i=1}^{n} i - \frac{3}{n} \sum_{i=1}^{n} 1 \\
&= \frac{27n(n+1)(2n+1)}{6n^3} + \frac{9n(n+1)}{2n^2} - 3 \\
&= 9 + \frac{27}{2n} + \frac{9}{2n^2} + \frac{9}{2} + \frac{9}{2n} - 3 \\
&= \frac{21}{2} + \frac{18}{n} + \frac{9}{2n^2} \\
\Rightarrow \int_0^3 x^2 + x - 1 \, dx = \lim_{n \to \infty} U(f, P_n) &= \frac{21}{2}
\end{aligned}$$

B) See class notes and text.

VII. A) True. The sequence of partial sums is monotone increasing if $a_n \geq 0$. Hence if s_N is bounded above, that sequence converges.

B) False. The series is absolutely convergent, since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3 > 1$. Conditionally convergent series are ones where $\sum_{i=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

C) True. Let x_n be any sequence with $\lim_{n \to \infty} x_n = 0$. Then whether or not x_n is rational, we have $\lim_{n \to \infty} f(x_n) = 0 = f(0)$. This shows $\lim_{x \to 0} f(x) = f(0)$, so f is continuous at $x = 0$.