Mathematics 242 – Principles of Analysis
Discussion 3 – The Derivative
November 10, 2004

Background

We have now developed the properties of limits and continuous functions. Our next topic will be a consideration of the foundations of differential calculus, starting with the concept of the derivative of a function. As in calculus, we say that a function $f : D \to \mathbf{R}$ is *differentiable* at $x = c$ if the limit

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \tag{1}$$

exists (the second form comes by introducing h as the difference between x and c and writing $x = c+h$). If so, we call the value of the limit the *derivative of* f *at* $x = c$, written $f'(c)$.

Note: In calculus, you learned a number of rules (the Product, Quotient, Chain Rules, and others) for computing derivatives. In some of the questions below, you will need to use these even though we have not developed them in this course. If the question says use the definition to compute $f'(c)$, though, you will need to use (1).

Discussion Questions

A)
1) Show using the definition of the derivative that for all positive integers n, the function $f(x) = x^n$ is differentiable for all c, and find $f'(c)$. Note: To calculate the limit for $f'(c)$, you will have to decide which form of the definition you want to use. If you use the second (the $\lim_{h\to 0}$ form), you will want to use the binomial theorem (Exercise 10.18). No matter which form you use, use the Limit Theorems for function limits (Theorem 20.13); don't go back to the ε-δ definition of the limit of a function.
2) Show using the definition of the derivative that $f(x) = \sqrt{x}$ is differentiable at all $c > 0$, and find $f'(c)$.

B) By considering the equation

$$f(x) = \frac{f(x) - f(c)}{x - c}(x - c) + f(c)$$

show that if f is differentiable at $x = c$, then f is continuous at $x = c$.

C)
1) Let

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is continuous at $c = 0$, but $f(x)$ is not differentiable at $c = 0$, by using the definition.

2) Let

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that g is differentiable at $c = 0$. Show using the Product and Chain Rules from calculus that $g'(x)$ exists for all $x \neq 0$ as well. Is $g'(x)$ a continuous function, though?

D) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

1) Show that $f(x)$ is continuous at $c = 0$, but $f(x)$ is not continuous at any $c \neq 0$.
2) Is $f(x)$ differentiable at $c = 0$? Why or why not?

E) Recall that a function $f : \mathbf{R} \to \mathbf{R}$ is said to be *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$ for all x.

1) Assume that f is differentiable for all x and even. What can you say about $f'(x)$? Prove your assertion.
2) Assume that f is differentiable for all x and odd. What can you say about $f'(x)$? Prove your assertion.

Assignment

Group write-ups due Monday, November 15.