1 Introduction

Textbook presentations of mathematics itself and histories of the subject unfortunately tend to suffer from some similar defects. In most mathematics textbooks, everything is seemingly inevitably and tidily organized. The actual process by which humans discover new mathematics (even something new only to a student) rarely, if ever, comes through; the focus is often on acquiring cut-and-dried techniques and mastering well-known algorithms. Analogously, until quite recently, histories have often presented aspects of the mathematical past as leading inevitably and tidily to our current understanding of the subject. Modern concepts and techniques have been freely applied to interpret original sources; little effort has been made to meet that past on its own terms or to understand the sometimes tortuous process by which mathematics has developed.

I have long wondered whether a better understanding of the human side of mathematics as revealed through its history might provide tools to improve the teaching of mathematics and change the way the subject is perceived by non-experts. And so, because the Greek mathematical tradition has been so important for the way mathematics has developed in the West, for the past four years, I have been learning ancient Greek – an intellectual adventure that I wish could be more common in today's secondary and post-secondary education.[1] My motivation for this was a desire to read works such as Euclid's

[1] I think the mental discipline and concentration needed even to begin to tackle a language with such an involved grammar, such a long history, and such a huge vocabulary are salutary in and of themselves. I would say that mental discipline and concentration are comparable to the habits of mind needed to do research in mathematics, even though there is not quite the same opportunity to exploit insight and creativity. Greek is supremely logical in its way, but many things must simply be committed to memory. Nevertheless I have experienced immense pleasure along the way and I would recommend such a course to anyone.

Elements and the *Conics* of Apollonius (or at least the portions of Apollonius that survive in Greek) in their original forms, guided by the recognition that all translations misrepresent their sources to some degree. Moreover, the mathematics of the past often starts from completely different assumptions and uses different methods that may not be captured in modern translations or expositions. Formally, I have been a special student in courses taught by several very kind colleagues in my home institution's well-regarded Classics department. All of them, by the way, say that really mastering ancient Greek is the work of a lifetime and I would certainly not claim to have done that. Yet I have read large chunks of those texts now and as a result I find my scholarly interests changing and my interest in the history of our discipline growing substantially.

In this essay I want to present a "report from the field," so to speak, describing a recent encounter with two well-known passages, one from one section of Plutarch's so-called *Moralia*, the compendium of occasional essays and miscellaneous writings that accompanies the series of parallel *Lives* of illustrious Greeks and Romans in his immense output, and the other from his *Life of Marcellus*.[2] These passages touch on a crucial episode in the history of the Greek mathematics of the fourth century BCE that stimulated mathematical research into the 19th century CE and whose influence is still felt at one point in the undergraduate pure mathematics curriculum–the story of various approaches to the duplication of the cube.[3]

[2] This essay originated in an assignment in Professor Thomas Martin's Plutarch seminar at Holy Cross in fall 2016 and I want to thank him for several very helpful comments and suggestions.

[3] This problem is usually grouped together with two others, the quadrature of the circle (i.e. the problem of constructing a square or rectangle with the same area as a given circle), and the trisection of a general angle. Wilbur Knorr examined this tradition of geometric problems in detail in [K1]. These passages from Plutarch have been seen by some (see [S] for a discussion) as a major contributing factor to the later notion that in Greek

One theme will be the way some key sources come from texts that have much wider cultural contexts and resonances. Sensitivity to the history, to the mathematics, and to the language is necessary to tease out their meanings. Yet, historians of mathematics have often interpreted these sources using the mathematics of their own times and produced what we can now see are questionable conclusions. Unfortunately it is those sometimes anachronistic accounts that have been presented in mainstream histories of mathematics to which mathematicians who do not read Greek must turn to learn about that history. At the same time, some classicists lack the mathematical background knowledge to appreciate that aspect of the content of these sources. Another theme is that once one returns to the actual source material, questions with no easy answers often abound and it is amazing to see how much we still do not understand about critical junctures in the history of our subject. The tidy and inevitable picture of the development of mathematics disappears and we are left with a much more interesting, if ultimately somewhat inconclusive, story.

Because this period and the associated questions have been intensively studied since the 19th century CE, I cannot claim that this essay presents any new historical scholarship. However, I hope that it may prove useful to instructors and other readers who are interested in finding more nuanced accounts of some of the Greek work on the duplication of the cube than are available in some of the standard histories.

geometric constructions, *only* the compass and straightedge were acceptable tools. In fact constructions using auxiliary curves of various sorts were developed for the quadrature of the circle and the trisection of general angles as well. Whether these constructions could be accomplished using only the Euclidean tools remained an open question until the work of P. Wantzel and others in the 19th century CE, completing an line of thought initiated by Descartes. It is now known that *none* of them is possible under those restrictive conditions. Many undergraduate mathematics majors learn proofs of these facts in abstract algebra courses.

2 The Plutarch Texts

Plutarch of Chaeronea (ca. 45 - ca. 120 CE) was a Greek writing for a mixed Greek and Roman audience during the early empire. He himself records that he studied philosophy and mathematics in Athens and his writings reveal a strong connection with Platonic traditions. We would call him an essayist and biographer, although most of his writing is more devoted to ethical lessons than to history, *per se*. The first passage I will discuss comes from a section of his *Moralia* known as the *Quaestiones Convivales*, or "Table Talk." Each section of this work is presented as a record of conversation at a *sumposion*, or drinking party, arranged by Plutarch for a group of guests. Philosophical questions are always debated and it is amusing to see what Plutarch says about the rationale for this: in "... our entertainments we should use learned and philosophical discourse ..." so that even if the guests become drunk, "... every thing that is brutish and outrageous in it [i.e. the drunkenness] is concealed ... ."[4] In other words, to keep your next party from degenerating into a drunken brawl, have your guests converse about Plato!

In Book 8, Chapter 2, Section 1 (classicists refer to this via the so-called Stephanus page 718 ef, from one of the first modern printed versions of the Greek text), Plutarch presents a conversation between the grammarian Diogenianus and the Spartan Tyndares concerning the role of the study of geometry in Plato's thought. Diogenianus begins this phase of the conversation by raising the question why Plato asserted that "God always geometrizes." He also says he is not aware of any specific text where Plato said precisely that, though he thinks it sounds like something Plato would have said. Tyndares replies that there is no great mystery there and asks Diogenianus whether it was not true that Plato had written that geometry is "... taking us away from

[4][QC], 716g, Book 8, Chapter 0, Section 2

the sensible and turning us back to the eternal nature we can perceive with our minds, whose contemplation is the goal of philosophy ... ."[5] Tyndares is evidently thinking of passages like 527b in Book VII of the *Republic*, where Plato has Socrates say in reference to geometry, "... it is the knowledge of that which always is ... it would tend to draw the soul to truth, and would be productive of a philosophical attitude of mind, directing upward the faculties that are now wrongly turned downward."[6]

After some elaboration of these points, Tyndares presents an interesting piece of evidence concerning this aspect of Plato's thought: "Therefore even Plato himself strongly criticized Eudoxus, Archytas, and Menaechmus" (or possibly "those around Eudoxus, Archytas, and Menaechmus") "for attempting to reduce the duplication of the cube to tool-based and mechanical constructions ... ."[7] Tyndares continues in a somewhat technical vein, "... just as though they were trying, in an unreasoning way, to take two mean proportionals in continued proportion any way that they might ... ."[8] It is quite interesting that Tyndares seems to be assuming that all of his listeners would be familiar with this terminology and the episode in the history of geometry to which he is referring. Tyndares concludes his summary of Plato's criticism by claiming that in this way (i.e. by using mechanical procedures

[5]In the text, I will present my translations of the Greek. Here: "ἀποσπῶσαν ἡμᾶς προσισχομένους τῇ αἰσθήσει καὶ ἀποστρέφουσαν ἐπὶ τὴν νοητὴν καὶ ἀίδιον φύσιν, ἧς θέα τέλος ἐστὶ φιλοσοφίας ... ." All Greek quotations from the Plutarch passage are from the Loeb Classical Library/Perseus online text of Plutarch's *Quaestiones Convivales*, [QC].

[6]English translation from [P], p. 758.

[7]"δίο καὶ Πλάτων αὐτὸς ἐμέμψατο τοὺς περὶ Εὔδοξον καὶ Ἀρχύταν καὶ Μέναιχμον εἰς ὀργανικὰς καὶ μηχανικὰς κατασκευὰς τὸν τοῦ στερεοῦ διπλασιασμὸν ἀπαγεῖν ἐπιχειροῦντας ... "

[8]"ὥσπερ πειρωμένους δι᾽ ἀλόγου δύο μέσας ἀνάλογον, ᾗ παρείκοι, λαβεῖν ... " The δι᾽ ἀλόγου is hard to translate and may not even be what Plutarch originally wrote. This specific phrase has a rather large number of textual issues as evidenced by the variant readings discussed in the Loeb Classical Library/Perseus version of the Plutarch text. At least one editor has suggested that the whole phrase should be omitted from the text!

with tools) "... the good of geometry is utterly destroyed and it falls back on things of the senses; it is neither carried above nor apprehends the eternal and immaterial forms, before which God is always God."[9] Tyndares is saying that Plato criticized the nature of the solutions proposed by Eudoxus, Archytas, and Menaechmus because they in effect subverted what he (i.e. Plato) saw as the true purpose of geometry: its *raison d'être* was not merely to solve problems "by any means necessary," but rather to lead the soul to the contemplation of eternal truth.[10]

It is interesting to note that Plutarch gives a second, partially parallel account of this criticism in Chapter 14, Sections 5 and 6 of his *Life of Marcellus*, in the context of a discussion of the geometrical and mechanical work of Archimedes and the tradition that King Hiero of Syracuse persuaded him to take up mechanics to design engines of war in defence of his native city-state. Marcellus was, of course, the commmander of the Roman forces in the siege of Syracuse in 212 BCE during which Archimedes was killed. In that passage, Plutarch says[11]

> "For the art of mechanics, now so celebrated and admired, was first originated by Eudoxus and Archytas, who embellished geometry with subtleties, and gave to problems incapable of proof by word and diagram, a support derived from mechanical illustrations that were patent to the senses. For instance in solving the problem of finding two mean proportional lines, a necessary

[9] "ἀπόλλυσθαι γὰρ ὀύτω καί διαφείρεσθαι τὸ γεωμετρίας ἀγαθὸν αὖθις ἐπὶ τὰ αἰσθητὰ παλινδρομούσης καὶ μὴ φερομένης ἄνω μηδ΄ ἀντιλαμβανομένης τῶν ἀίδιων καὶ ἀσωμάτων εἰκόνων πρὸς αἷσπερ ὢν ὁ θεὸς ἀεὶ θεός ἐστι."

[10] In his thought-provoking study of the Greek work on these construction problems, [K1], Wilbur Knorr has argued in effect that by this period Greek mathematics had distanced itself from the sort of philosophical or religious underpinning that Plutarch has Tyndares say Plato claimed for it and was already very close to a modern research program, in which the goal *often is indeed* at to solve problems by whatever means are necessary, and then try to understand what methodological restrictions might still allow a solution. (See the discussions on pp. 39–41 and p. 88 in particular.) Needless to say, this view is not universal among historians of Greek mathematics.

[11] (in the English translation by Bernadotte Perrin from [LM])

requisite for many geometrical figures, both mathematicians had recourse to mechanical arrangements[12] adapting to their purposes certain intermediate portions of curved lines and sections.[13] But Plato was incensed at this, and inveighed against them as corrupters and destroyers of the pure excellence of geometry, which thus turned her back upon the incorporeal things of abstract thought and descended to the things of sense, making use, moreover, of objects which required much mean and manual labor. For this reason, mechanics was made entirely distinct from geometry, and being for a long time ignored by philosophers, came to be regarded as one of the military arts."

As we will see, these passages in Plutarch provide a fascinating, but also ultimately somewhat cryptic, sidelight on a key episode in the history of Greek mathematics.

3 The Historical Background

Eudoxus of Cnidus (409–356 BCE), Archytas of Tarentum (428–347 BCE), and Menaechmus of Alopeconnesus (380–320 BCE) were three of the most accomplished Greek mathematicians active in the 4th century BCE. Archytas is often identified as a Pythagorean and there are traditions that Eudoxus was a pupil of his and Menaechmus was a pupil of Eudoxus. All three were associated with Plato and his Academy in Athens in some way.[14]

As almost all mathematicians know, the *duplication of the cube* was a geometrical problem asking for the construction of the side of a cube whose

[12] "κατασεκυάς" – "constructions" would be another, perhaps better, translation here.

[13] "μεσογράφους τινὰς ἀπὸ καμπύλων καὶ τμημάτων μεθαρμόζοντες." I think a better translation here is "adapting to their purposes mean proportionals found from curved lines and sections." The meaning of μεσογράφους here is the same as in a passage from Eratosthenes that will be discussed below. One can see the dilemma of a non-mathematician translating technical discussions! In the passage as a whole, one can also glimpse some of the less attractive aspects of Plato's thought.

[14] We have much of this from sources such as Proclus, [P], pp. 54–56, though the fact that Proclus is writing roughly 800 years after this period raises the question of how reliable his information is.

volume would be twice the volume of a given cube. Various traditions deal with the genesis of this problem. One says that seeking direction in order to stem the progress of a plague on their island (or perhaps political conflicts; different versions of the story differ on this point), the people of Delos consulted the oracle at Delphi, whereupon the Pythia replied that they must find a way to double the size of an cubical altar of Apollo.[15] When they were unable to do this themselves, the Delians supposedly consulted Plato and the geometers at his Academy to find the required geometric construction; for this reason the problem of the duplication of the cube is often called the "Delian problem." However, this version of the story is almost certainly fanciful (at least as the *origin* of the problem–the chronology seems to be wrong, for one thing, since as we will see presently there was work on the question somewhat before the time of Plato (428 – 348 BCE)). Plutarch himself, in another section of the *Moralia* called *The E at Delphi*, says that the underlying point of the story was that the god was commanding the Greeks to apply themselves more assiduously to geometry.[16] There is no doubt that the duplication of the cube was one of a series of geometric construction problems that stimulated the development of Greek mathematics throughout the Classical period.

For a full understanding of our Plutarch passages, and of Plato's supposed objection to the work of Eudoxus, Archytas, and Menaechmus, we need to introduce an important piece of progress that had been made earlier and definitely before the time of Plato by Hippocrates of Chios (ca. 470–ca. 410 BCE). None of Hippocrates' own writings have survived and we know

[15]A somewhat parallel story about King Minos seeking how to double the size of a tomb also appears in a letter of Eratosthenes to King Ptolemy III Euergetes of Egypt that will be discussed below. See [H], p. 245.

[16][DeE], Chapter 6, 386e.

about the following only from sources such as fragments of a history of pre-Euclidean mathematics by Eudemus of Rhodes (ca. 370 – ca. 300 BCE) preserved in other sources. Given two line segments AB and GH, we say line segments CD and EF are *two mean proportionals in continued proportion*[17] between AB and GH if their lengths satisfy:

$$\frac{AB}{CD} = \frac{CD}{EF} = \frac{EF}{GH}. \tag{1}$$

Hippocrates' contribution was the realization that if we start with

$$GH = 2AB,$$

then the construction of two mean proportionals as in (1) would *solve the problem of the duplication of the cube.* The idea is straightforward: If

$$\frac{AB}{CD} = \frac{CD}{EF} = \frac{EF}{2AB},$$

then some simple algebra (which the Greeks would have emulated with parallel manipulation of proportions) shows

$$CD^3 = 2AB^3.$$

In other words, if AB is the side of the original cube, then CD is the side of the cube with twice the volume. With this observation, Hippocrates did not give a full solution for the duplication of the cube, but he did provide a way to attack the problem. Almost all later work took his reduction to the construction of the two mean proportionals as its starting point.

[17]In the first Plutarch passage above, this appears in the accusative as δύο μέσας ἀνάλογον. The ἀνάλογον seems to be essentially equivalent to the ἀνὰ λόγον from the second passage, and that is conventionally translated as "in continued proportion."

4 Do We Know What Eudoxus, Archytas, and Menaechmus Did?

Plutarch does not include any discussion of what Eudoxus, Archytas, or Menaechmus actually did in their work on the duplication of the cube. However, historical accounts of the work on this problem including information about their approaches have survived in ancient sources. The most important is a much later commentary on Archimedes' *On the Sphere and the Cylinder* by Eutocius of Ascalon (ca. 480 – ca. 540 CE), in which Eutocius surveys a wide selection of different solutions to the problem of duplicating a cube by finding two mean proportionals, as discussed in our Plutarch passages.[18] Eutocius includes in his account a purported letter to King Ptolemy III of Egypt by Eratosthenes of Cyrene (276 – 194 BCE) with a summary of earlier work and Eratosthenes' own, definitely mechanical and tool-based, solution making use of an instrument dubbed the *mesolabe*, or "mean-taker."[19]

I will discuss the main ideas behind what Eutocius says about the approaches of Archytas and Menaechmus and the ways this evidence has been interpreted. The solution by Eudoxus is not presented in detail by Eutocius because he believes his sources for it are corrupt. Hence we do not have enough information to draw any conclusions about how it connects with what

[18] The occasion for the commentary was the fact that Archimedes assumed the construction was possible in some way in the proof of the first proposition in Book II of *On the Sphere and the Cylinder*, but he did not provide any explanation. Because of the nearly 1000 years intervening between the time of the Platonic geometers and Eutocius' time, the caveat we made above about accepting evidence from Proclus' writings uncritically also applies to Eutocius' writings. The Greek original with a Latin translation is included in Volume III of J. Heiberg's *Archimedis Opera Omnia*, [A]. A near-literal English translation is given by Netz in [N].

[19] The purpose of the letter is essentially to claim the superiority of Eratosthenes' tool-based mechanical method for practical use. It was dismissed as a later forgery by some 19th and early 20th century historians, but more recently, the tide of opinion has seemingly changed and sources such as [K1] argue that it should be accepted as authentic.

Plutarch writes. My goals here are to shed some additional light on the content of the Plutarch passages and to show how different interpretations of what Eutocius says have led to quite different understandings of this part of Greek geometry. Some of these seem more faithful to context of this work and some seem more anachronistic.

To begin, we should discuss how "instrument-" or "tool-based"[20] and "mechanical"[21] or might apply to geometric constructions. On the face of it, "instrument-based," or "tool-based," is clearer. For that adjective to apply, I believe some physical device such as the *mesolabe* of Eratosthenes must be involved in the construction. But even there, there is a point that is subtle for some modern readers of these works. The Greeks, even though they used physical straightedges to draw lines and physical compasses to draw circles while constructing diagrams, also considered those tools in *idealized versions* that were constructs of the mind and not dependent on the senses. The first three postulates in Book I of the *Elements* of Euclid (ca. 300 BCE) describe their uses and properties in abstract terms. In particular, the idealized straightedge can be used to draw lines, but not to measure distances; it has no distance scale like a modern ruler. Moreover, the idealized Euclidean compass can be used to draw circles, but not to measure or transfer distances. So we should certainly not take criticisms such as the one ascribed to Plato here to refer to constructions that involve the Euclidean tools.

What counts as "mechanical" is not entirely clear either. For Plutarch, from the evidence of the discussion in the *Life of Marcellus*, it seems that the adjective "mechanical" referred to the use of mathematics to design and emulate the motions of real-world machines, and perhaps machines of war in particular. It is possible that for him the adjectives "tool-based" and

[20] ὀργανικός
[21] μηχανικός

"mechanical" overlap in meaning to some degree. Hence, I would suggest that another aspect that might make a geometrical construction "mechanical," apart from the use of actual tools, is that it has some element of *motion* (real or imagined) as in a real-world mechanical device. Note that any sort of change over time in a figure would by itself seem to violate Plato's vision of the eternal and unchanging nature of the world of the forms.

5 The Work of Archytas

In Eutocius' presentation, the account of the approach by Archytas is specifically attributed to Eudemus' history, which means what we have may actually be a commentary on a commentary.[22] The solution is essentially based on a geometric configuration in which it can be seen that two mean proportionals in continued proportion have been found. Borrowing from [M], we will call these *Archytas configurations*. One of these is shown in Figure 1. Here AEB and ADC are two semicircles tangent at A, and BD is tangent to the smaller semicircle at B. It follows from some standard geometric facts that ΔBAE, ΔCAD, ΔDBE, ΔCDB and ΔDAB are all similar. This follows because the angles $\angle AEB$ and $\angle ADC$ are inscribed in semicircles, hence right angles, and hence the lines $\overleftrightarrow{EB}$ and $\overleftrightarrow{DC}$ are parallel. From this we can see immediately that taking ratios of longer sides to hypotenuses in three of these triangles,

$$\frac{AE}{AB} = \frac{AB}{AD} = \frac{AD}{AC}.$$

In other words, AB and AD are two mean proportionals in continued proportion between AE and AC.

But now, we must address the question of how such a configuration would

[22] The other sections of Eutocius' commentary are not labeled in this way; they give the name of the author, and sometimes the title of the work from which Eutocius is quoting.

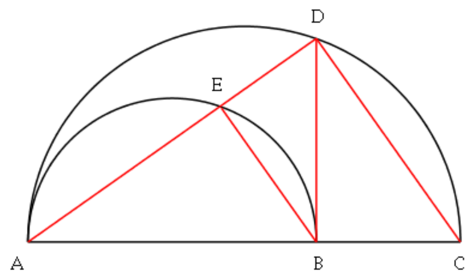

Figure 1: An *Archytas configuration*

be constructed given the lengths $AE < AC$. A modern explanation might run as follows. The issue is that although we can always take the segment AC as the diameter of the larger semicircle, there is no direct way to construct the smaller semicircle, the perpendicular BD to AC and the point E without some sort of continuity argument or approximation process. Consider the situation in Figure 2. Given the lengths $AE < AC$, the possible locations of the point E lie on an arc of the circle with center at A and radius equal to a specified length. One such arc is shown in blue in Figure 2. Through each point E on that arc, there is exactly one semicircle tangent at A to the semicircle with diameter AC, shown in green in the figure. The line through A, E meets the outer semicircle at D and B is the foot of the perpendicular from D to AC. However, note that with this choice of E, DB *does not meet the smaller semicircle at all.* However, by rotating the segment AE about A and increasing the angle $\angle CAE$, we would eventually find that the corresponding BD cut through the corresponding smaller semicircle. Hence there must be

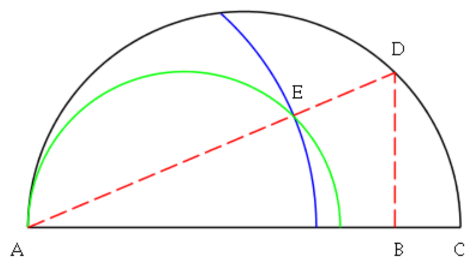

Figure 2: A failed attempted construction of an *Archytas configuration*

some point E on the blue arc that yields an Archytas configuration as in Figure 1, by continuity. As we have described it, a naive process of finding that point might involve motion and exactly the sort of resort to "eyeballing" or use of the senses that Tyndares says Plato criticized in our passages from Plutarch![23]

What Eutocius said that Archytas actually did here has been interpreted in a number of different ways by different modern scholars. One tradition known from influential sources such as T. L. Heath's *A History of Greek Mathematics*, [H], interprets Archytas' solution as a bold foray into solid geometry whereby a suitable point like our E in Figure 1 is found by the intersection of three different surfaces in three dimensions (a cylinder, a cone and a degenerate semi-torus–the surface of revolution generated by rotating the semicircle with diameter AC about its tangent line at A).[24]

[23]We can also easily locate such a point using modern coordinate geometry, trigonometry, and numerical root finding. But needless to say, all of that is well beyond the scope of Greek mathematics.

[24]See for instance [H], pp. 246–249. Heath characterizes this solution as "the most

Very recently, however, a new interpretation based on a close reading of the Greek text of Eutocius has appeared in the historical literature in [M]. As the author Masià points out, it is not easy to see all of the aspects of Heath's description in the actual text. While a cylinder and a cone are explicitly mentioned, the semi-torus surface of revolution is not. Moreover, even there, the cone and its properties are not really used in the proof; it seems to be included more for the purposes of visualization.[25] Hence Heath's interpretation, while certainly a correct way to describe the geometry, seems to be an anachronistic reading.

Instead, Masià suggests that the argument can be understood in a fashion that seems much closer to what we know about the state of geometry at the time of Archytas. In a preliminary step, an Archytas configuration is found by starting from a semicircle and an inscribed triangle two of whose sides have the given lengths AE and AC. A second copy of the semicircle and a moving inscribed triangle are rotated about A until an Archytas configuration is reached. This motion is then emulated in three dimensions by triangles in two perpendicular planes to produce a construction matching the Greek text of Eutocius very closely.[26] In either our simple presentation, or the kinematic description of the three surfaces in three dimensions, or the new reading of Archytas' construction from [M], there is definitely an aspect of motion that seems to agree with Plato's reported characterization of the construction

remarkable of all" because of the sophisticated use of three-dimensional geometry he sees in it. Similarly, Knorr calls it a "stunning *tour de force* of stereometric insight" in [K1], p. 50.

[25][M], p. 203.

[26]See [M], pp. 188–193. Masià discusses several other possible ways to interpret Archytas' solution in two or three dimensions and discusses other interpretations including the one given in [K3], Chapter 5. He also points out that Heath's characterization of this solution as the "most remarkable" does not seem to match the way the solution is presented by Eutocius.

as "mechanical."[27] How the adjective "instrument-" or "tool-based" might apply is not as clear, although one could easily imagine a device to carry out the planar version of the construction given in Figure 11 of [M].

6 The Work of Menaechmus

The approach attributed by Eutocius to Menaechmus is even more problematic although it was evidently extremely influential for the development of a key part of Greek geometry. This approach can be described (very anachronistically) as follows.[28] Given line segments of lengths a, b, finding the two mean proportionals in continued proportion means finding x, y to satisfy:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}. \tag{2}$$

Hence, cross-multiplying and interpreting the resulting equations via coordinate geometry, we see the solution will come from the point of intersection of the parabola $ay = x^2$ and the hyperbola $xy = ab$, or one of the points of intersection of the two parabolas $ay = x^2$ and $bx = y^2$. A related piece of evidence is the epigram of Eratosthenes on the duplication of the cube that concludes the letter to Ptolemy III mentioned above. This includes the direction "neither seek to cut the cone in the triads of Menaechmus" to obtain a solution.[29]

[27]On the other hand, as Masià points out on p. 197 of [M], the third person perfect imperative verb forms typically used in Greek to mark the steps of geometric constructions (e.g. γεγράφθω – "let it have been drawn") seem to emphasize that the figure or diagram *has been constructed* as a whole, and thus connote something static rather than something dynamic. This convention seems in fact to agree perfectly with the Platonic conception of geometry that forms the basis of the criticism in our Plutarch passages. One can imagine Archytas responding to Plato's criticism by pointing out that there is no actual motion involved!

[28]This is essentially the presentation given in [H], pp. 252–255, although Heath does not use coordinate geometry explicitly in this way.

[29]"μηδὲ Μεναιχμείους κωνοτόμειν τριάδας διζήαι ... .," [A], p. 112. (The present middle indicative διζήαι should probably be the aorist middle subjunctive διζήσῃ and other sources

For these reasons, Menaechmus has often been credited with initiating the study of the *conic sections*, later taken up and elaborated by Euclid, Archimedes, Apollonius, and other Greek mathematicians. However, it must be said that although this claim has been repeated in a number of standard histories of mathematics, including [H] and [BM][30], the case that Menaechmus did any more than to stimulate those later developments on conic sections with his work on the Delian problem is flimsy at best. It rests more on imaginative reconstructions of his "probable methods" produced by Zeuthen, Coolidge, and Heath in the 19th century than on any direct evidence.[31]

None of Menaechmus' own writings have survived and (even more suspiciously) the discussion of his work in Eutocius uses the terminology for conic sections introduced much after the time of Menaechmus himself by Apollonius of Perga (262–190 BCE).[32] Earlier terminology, according to which parabolas are "sections of right-angled cones" (by planes perpendicular to one of the generating lines of the cone) and hyperbolas are "sections of obtuse-angled cones" is preserved in such works as Archimedes' *Quadrature of the Parabola*. This is sometimes used to infer the way Menaechmus may have

correct it that way.) See for instance [H], p. 246. Note that Eratosthenes was active significantly later than Menaechmus himself and the initial steps in the elaboration of the theory of conic sections would have intervened. Exactly what this phrase means is also not entirely clear. Some writers have seen in the "triads" the division of conic sections into the three classes of ellipses, parabolas, and hyperbolas. However to the extent that it applies to Menaechmus' work at all, this is surely anachronistic; it seems more likely that the "triads" refer essentially to the three curves obtained from the proportionality relations in (2).

[30] [H], pp. 251–255; [BM], pp. 84–87.

[31] My thinking on this has been strongly influenced by [K2] and the discussion in [K1], pp. 63–69.

[32] Since that terminology seems to have been developed by analogy with constructions in the application of areas, there is no doubt that some connection between Menaechmus and the later theory of conics exists. The point I wish to make here is that it seems anachronistic to attribute this discussion, *in the form given by Eutocius*, to Menaechmus himself.

approached the definition of the conics as well as the way that theory may have been presented in the lost *Conics* of Euclid. On the other hand, in [K1], pp. 63–69, Knorr presents an alternative conjectural reconstruction of Menaechmus' work that does not rely on curves derived as sections of cones. His version is also at least plausible. In any case, it seems likely that either a source Eutocius consulted or Eutocius himself reworked Menaechmus' presentation in the light of later developments.

Unfortunately, with our fragmentary knowledge from the surviving ancient sources, we cannot really be sure about any of this. I would venture, though, that attributing a full-blown theory of conic sections (that is, as plane sections of cones) to Menaechmus may be yet another instance of the sort of *conceptual anachronism* that unfortunately abounds in conventional and accepted histories of mathematics.[33]

One of the slightly mysterious aspects of the Platonic criticism recounted in our Plutarch passages is how the adjectives "mechanical" or "instrument-" or "tool-based" might apply to what is attributed to Menaechmus by Eutocius. It is true that the conic sections apart from the circle cannot be constructed as whole curves using only the Euclidean tools and other sorts of devices would be needed to produce them. Interestingly enough, along these lines, Eutocius' discussion does include a final comment that "the parabola is drawn by the compass invented by our teacher the mechanician Isidore of Miletus ... ."[34] Isidore (442–537 CE) was an architect, one of the designers of the *Hagia Sophia* in Constantinople, and thus this note is surely an inter-

[33] This may in fact apply just as much to Eutocius as to modern historians of mathematics. In my opinion, mathematicians often make bad historians. They don't always distinguish between logically equivalent forms of statements and they tend to attribute their own understanding of those statements to mathematicians of the past.

[34] "γράφεται δὲ ἡ παραβολὴ διὰ τοῦ εὑρεθέντος διαβήτου τῷ Μιλησίῳ μηχνικῷ Ἰσιδώρῳ τῷ ἡμετέρῳ διδασκάλῳ ... ", [A], p. 98.

polation from the general period of Eutocius himself, not a part of the older source Eutocius was using to produce this section of his commentary.[35]

7 Plato's Solution?

In a final, decidedly odd, aspect of this story, Eutocius also gives a construction of the two mean proportionals in continued proportion that he ascribes to Plato himself. But that is one of the most mechanical and tool-based of all the solutions he describes in that it requires the use of a frame something like two "t-squares" joined along one edge.[36] The configuration containing the two mean proportionals is found by maneuvering the device until one point is found to coincide with one endpoint of one of the given segments.[37] A sort of *neusis*, or limiting process, requring input from the senses of the geometer is the crucial component. As Knorr says, "one is astounded at the flexibility of the traditions which on the one hand attribute such a mechanism to Plato, yet on the other hand portray him as the defender of the purity of geometry and the sharp critic of his colleagues for their use of mechanical procedures in geometric studies."[38]

8 Conclusions

Our study of the passages in question shows that Plutarch has seemingly preserved a largely accurate picture of Plato's thinking, certainly more accurate than some of the traditions preserved in Eutocius' commentary. But from what we know of the work of Archytas and Menaechmus and from the later

[35]See [N], p. 290, note 130.
[36]The text discussion is accompanied by a rare perspective drawing of the device.
[37]There is a good diagram of this on page 58 of [K1].
[38][K1], p. 59.

work of Archimedes, Apollonius and others, I would argue that if something like Plato's criticism of the geometers in his circle actually happened at this point in history, then its effect on Greek mathematics was rather minimal.

An openness to mechanical techniques can already be seen for instance in the description of the *quadratrix* curve ascribed to Hippias of Elis (late 5th century CE) and used in the period we have considered in solutions of the angle trisection and circle quadrature problems. We often find scholars of the Hellenistic and later periods pursuing both mathematical and mechanical work, sometimes even in combination. Celebrated examples of this trend include Archimedes' work on spirals, a portion of his *Quadrature of the Parabola*, and most strikingly his *Method of Mechanical Theorems*, which presents a somewhat systematic procedure, based on mechanics, to discover geometric area and volume mensuration results.[39] Even later, Heron of Alexandria (ca. 10 – ca. 70 CE) gives another different solution of the problem of the two mean proportionals in his Βελοποιίκα, a treatise on the design of siege engines and artillery(!)

While it drew on philosophy for its norms of logical rigor, I would agree with Knorr that mathematics had in essence emerged as an independent subject in its own right by the time of Eudoxus, Archytas, and Menaechmus. Plutarch was, by training and inclination, a Platonist and this by itself sufficiently explains his interest in preserving traditions about Plato's thinking about mathematics and his criticism of the geometers in his circle. Yet it is doubtful that Plato's ideas about the proper methods or goals of mathematics carried much real weight for many of the actual practitioners of the

[39] Archimedes uses dissection procedures akin to the subdivisions used in modern integral calculus, combined with an idealized balance beam. Most modern mathematicians would probably even be happy to consider his arguments as complete proofs, although Archimedes himself had scruples about that point.

subject.

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