Plato's Criticism of Geometers of His Time
From Plutarch, *Quaestiones Convivales*, 718 e-f (Book 8, Chapter 2, Section 1)

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In *Quaestiones Convivales*, 718 e-f (Book 8, Chapter 2, Section 1) Plutarch presents a conversation between the grammarian Diogenianus and the Spartan Tyndares concerning the role of the study of geometry in Plato's thought. Diogenianus begins this phase of the conversation by raising the question why Plato asserted that "God always geometrizes." He also says he is not aware of any specific text where Plato said precisely that, though he thinks it sounds like something Plato would have said. Tyndares replies that there is no great mystery there and asks Diogenianus whether it was not true that Plato had often written that geometry is "ἀποσπῶσαν ἡμᾶς προσισχομένους τῇ αἰσθήσει καὶ ἀποστρέφουσαν ἐπὶ τὴν νοητὴν καὶ ἀίδιον φύσιν, ἧς θέα τέλος ἐστὶ φιλοσοφίας ... ."[1] After some elaboration of these points, Tyndares presents an interesting piece of evidence concerning this aspect of Plato's thought: "δίο καὶ Πλάτων αὐτὸς ἐμέμψατο τοὺς περὶ Εὔδοξον καὶ Ἀρχύταν καὶ Μέναιχμον εἰς ὀργανικὰς καὶ μηχανικὰς κατασκευὰς τὸν τοῦ στερεοῦ διπλασιασμὸν ἀπαγεῖν ἐπιχειροῦντας ... ."[2] As we will see, this passage in Plutarch

[1] All Greek quotations are from the Loeb Classical Library/Perseus text of Plutarch's *Quaestiones Convivales*, [QC]. My translations will be given in footnotes like this one: "... taking us away from the sensible and turning us back to the eternal nature we can perceive with our minds, whose contemplation is the goal of philosophy ... ." Tyndares is apparently thinking of passages like 527b in Book VII of the *Republic*, where Plato has Socrates say in reference to geometry, "... it is the knowledge of that which always is ... it would tend to draw the soul to truth, and would be productive of a philosophical attitude of mind, directing upward the faculties that are now wrongly turned downward" (English translation from [P], p. 758).

[2] "Therefore even Plato himself strongly criticized Eudoxus, Archytas, and Menaechmus" (or possibly "those around Eudoxus, Archytas, and Menaechmus") "for attempting to reduce the *duplication of the cube* to mechanical constructions with instruments ... " Plutarch gives a second, largely parallel account of this criticism in Chapter 14 of his Life of Marcellus, in the context of a discussion of the geometrical and mechanical work of Archimedes and the tradition that King Hiero of Syracuse persuaded him to take up mechanics to design engines of war in defence of his native city-state. Marcellus was, of course, the commmander of the Roman forces in the siege of Syracuse in 212 BCE during which Archimedes was killed. In that passage, the ancient Greek distinction between mathematics and mechanics is laid out in even greater detail, and Eudoxus and Archytas are cited as originators of famous mechanical techniques.

provides a fascinating, but also ultimately somewhat cryptic, sidelight on a key episode in the history of Greek mathematics.

Eudoxus of Cnidus (409–356 BCE), Archytas of Tarentum (428–347 BCE), and Menaechmus of Alopeconnesus (380–320 BCE) were three of the most accomplished Greek mathematicians active in the 4th century BCE. Archytas is often identified as a Pythagorean and there are traditions that Eudoxus was a pupil of his and Menaechmus was a pupil of Eudoxus. All three were associated with Plato and his Academy in Athens in some way.[3]

The *duplication of the cube* was a geometrical problem asking for the construction of the side of a cube whose volume would be twice the volume of a given cube. Various traditions deal with the genesis of this problem. One says that seeking direction in order to stem the progress of a plague on their island (or perhaps political conflicts; different versions of the story differ on this point), the people of Delos consulted the oracle at Delphi, whereupon the Pythia replied that they must find a way to double the size of an altar of Apollo.[4] When they were unable to do this themselves, the Delians supposedly consulted Plato and the geometers at his Academy to find the required geometric construction; for this reason the problem of the duplication of the cube is often called the "Delian problem." Plutarch himself in *The E at Delphi* says that the underlying point of the story was that the god was commanding the Greeks to apply themselves to geometry.[5] However, this version of the story is probably fanciful (at least as the *origin* of the problem–the chronology seems to be wrong, for one thing, since as we will see

[3]We have much of this from sources such as Proclus, [P], pp. 54–56, though the fact that Proclus is writing roughly 800 years after this period raises the question of how reliable his information is.

[4]A somewhat parallel story about King Minos seeking how to double the size of a tomb also appears in a letter of Eratosthenes to King Ptolemy III of Egypt that will be discussed below. See [H], p. 245.

[5][DeE], Chapter 6, 386E.

presently there was work on the question somewhat before the time of Plato (428 – 348 BCE)). However, there is no doubt that the duplication of the cube was one of a series of geometric construction problems that stimulated much of the development of Greek mathematics throughout the Classical period.[6]

Tyndares continues in a somewhat technical vein, "ὥσπερ πειρωμένους δι᾽ ἀλόγου δύο μέσας ἀνάλογον, ᾗ παρείκοι, λαβεῖν ... ".[7] It is quite interesting that Tyndares seems to be assuming that all of his listeners would be familiar with this terminology and the episode in the history of geometry to which he is referring.

For a full understanding of this, and of Plato's supposed objection to the work of Eudoxus, Archytas, and Menaechmus, we need to introduce an important piece of progress that had been made toward the solution of the duplication of the cube problem earlier and definitely before the time of Plato, by Hippocrates of Chios (ca. 470–ca. 410 BCE). Given two line segments AB and GH, we say line segments CD and EF are *two mean proportionals in continued proportion*[8] between AB and GH if their lengths satisfy:

$$\frac{AB}{CD} = \frac{CD}{EF} = \frac{EF}{GH}. \tag{1}$$

[6] The others were the *quadrature of the circle* (that is, the problem of constructing a square equal in area to a given circle), and the *trisection of a general angle*. Wilbur Knorr examined this tradition of geometric problems in detail in [K].

[7] I propose this reading: "just as though they were trying, in an unreasoning way, to take two mean proportionals in continued proportion any way that they might ... ". The δι᾽ ἀλόγου is hard to translate and may not even be what Plutarch originally wrote. This specific phrase has a rather large number of textual issues as evidenced by the variant readings discussed in the Loeb Classical Library/Perseus version of the Plutarch text.

[8] In the Plutarch passage, this appears in the accusative as δύο μέσας ἀνάλογον. The ἀνάλογον seems to be essentially equivalent to ἀνὰ λόγον, and that is conventionally translated as "in continued proportion."

Hippocrates' contribution was the realization that if we start with

$$GH = 2AB,$$

then the construction of two mean proportionals as in (1) would *solve the problem of the duplication of the cube.* The idea is straightforward: If

$$\frac{AB}{CD} = \frac{CD}{EF} = \frac{EF}{2AB},$$

then some simple algebra (which the Greeks would have emulated with parallel manipulation of proportions) shows

$$CD^3 = 2AB^3.$$

In other words, if AB is the side of the original cube, then CD is the side of the cube with twice the volume. With this observation, Hippocrates did not give a full solution for the duplication of the cube, but he did provide a way to attack the problem. Almost all later work took his reduction to the construction of the two mean proportionals as its starting point.

Tyndares concludes his summary of Plato's criticism by claiming that by mechanical procedures with tools "ἀπόλλυσθαι γὰρ ... καί διαφείρεσθαι τὸ γεωμετρίας ἀγαθὸν αὖθις ἐπὶ τὰ αἰσθητὰ παλινδρομούσης καὶ μὴ φερομένης ἄνω μηδ᾽ ἀντιλαμβανομένης τῶν ἀίδιων καὶ ἀσωμάτων εἰκόνων πρὸς αἷσπερ ὢν ὁ θεὸς ἀεὶ θεός ἐστι."[9] Tyndares is saying that Plato criticized the nature of the solutions proposed by Eudoxus, Archytas, and Menaechmus because they in effect subverted what he saw as the true purpose of geometry: its *raison d'être* was not merely to solve problems "by any means necessary,"

[9] "... the good of geometry is utterly destroyed and it falls back on things of the senses; it is neither carried above nor apprehends the eternal and immaterial forms, before which God is always God."

but rather to lead the soul to the contemplation of eternal truth.[10]

Plutarch does not include any discussion of what Eudoxus, Archytas, or Menaechmus actually did to to find the two mean proportionals between two given line segments. However, historical accounts of the work on this problem including information about their approaches have survived in ancient sources. The chief one of these sources is a much later commentary on Archimedes' *On the Sphere and the Cylinder* by Eutocius of Ascalon (ca. 480 – ca. 540 CE), in which Eutocius surveys a wide selection of different solutions to the problem of finding the two mean proportionals.[11] Eutocius includes in his account a letter to King Ptolemy III of Egypt by Eratosthenes of Cyrene (276 – 194 BCE) with a summary of earlier work and Eratosthenes' own, definitely mechanical and tool-based, solution making use of an instrument he dubbed the *mesolabe*, or "mean-taker."[12]

I will present the main ideas behind what Eutocius says about the approaches of Archytas and Menaechmus in order to shed some additional light on the content of the Plutarch passage. The solution by Eudoxus is not presented in sufficient detail by Eutocius for us to form a definitive impression of how it connects with what Plutarch writes.

To begin, we should discuss how the adjectives μηχανικός or ὀργανικός

[10] But in his thought-provoking study of the Greek work on these construction problems, [K], Wilbur Knorr has argued in effect that by this period Greek mathematics had distanced itself from the sort of philosophical or religious underpinning that Tyndares says Plato claimed for it and was already very close to a modern research program, in which the goal *often is indeed* to solve problems by whatever means are necessary. This view is not universal among historians of Greek mathematics.

[11] The occasion for this was the fact that Archimedes assumed the construction was possible in some way in the proof of the first proposition in Book II of *On the Sphere and the Cylinder*, but he did not provide any explanation.

[12] The purpose of the letter is essentially to claim the superiority of Eratosthenes' mechanical method for practical use. It was dismissed as a later forgery by some 19th and early 20th century historians, but more recently, the tide of opinion has seemingly changed and sources such as [K] argue forcefully that it should be accepted as authentic.

might apply to geometric constructions. On the face of it, ὀργανικός, in the sense of instrument-based, or tool-based, is clearer. For the adjective ὀργανικός to apply, I believe some physical device such as the *mesolabe* of Eratosthenes must be involved in the construction. But even there, there is a slightly subtle point. The Greeks, even though they almost certainly used physical straightedges to draw lines and physical compasses to draw circles while constructing diagrams, were apparently also happy to consider those tools in *idealized versions* that were constructs of the mind and not dependent on the senses.[13] What counts as μηχανικός is unfortuately even less clear. For the purposes of this discussion, one possibility is a construction that has some element of *motion* (real or imagined) as in real-world mechanical devices. Note that any sort of change over time in a figure would by itself seem to violate Plato's vision of the eternal and unchanging nature of the world of the forms. Another sort of violation might occur in a construction that involves some sort of measurement.[14]

The approach by Archytas is essentially based on a geometric configuration in which it can be seen that two mean proportionals in continued proportion have been found. We will call these *Archytas configurations* for simplicity. One of these is shown in Figure 1. Here AEB and ADC are two semicircles tangent at A, and BD is tangent to the smaller semicircle at B. It follows from some standard geometric facts that ΔBAE, ΔCAD, ΔDBE, ΔCDB and ΔDAB are all similar. This follows because the angles $\angle AEB$

[13]The first three postulates in Book I of the *Elements* of Euclid (ca. 300 BCE) describe their uses and properties in abstract terms. In particular, the idealized straightedge can be used to draw lines, but not to measure distances; it has no distance scale like a modern ruler. Moreover, the idealized Euclidean compass can be used to draw circles, but not to measure or transfer distances.

[14]Both of these sorts of operations are clearly useful and even necessary in what we would today call *applied mathematics*, and it is exactly that aspect of Archimedes' work that Plutarch discusses in the passage from the Life of Marcellus mentioned above.

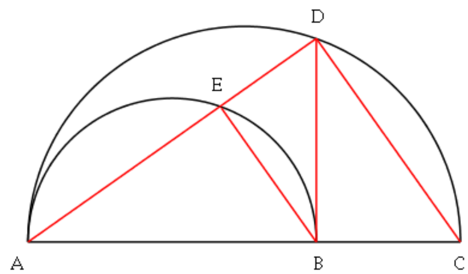

Figure 1: An *Archytas configuration*

and $\angle ADC$ are inscribed in semicircles, hence right angles, and hence EB and DC are parallel. From this we can see immediately that taking ratios of longer sides to hypotenuses in three of these triangles,

$$\frac{AE}{AB} = \frac{AB}{AD} = \frac{AD}{AC}.$$

In other words, AB and AD are two mean proportionals in continued proportion between AE and AC.

But now, we must address the question of how such a configuration would be constructed given the lengths $AE < AC$. The issue is that although we can always take the segment AC as the diameter of the larger semicircle, there is no direct way to construct the smaller semicircle, the perpendicular BD to AC and the point E without some sort of continuity argument or approximation process. A modern explanation might run as follows. Consider the situation in Figure 2.

Given the lengths $AE < AC$, the possible locations of the point E lie on an arc of the circle with center at A and radius equal to a specified

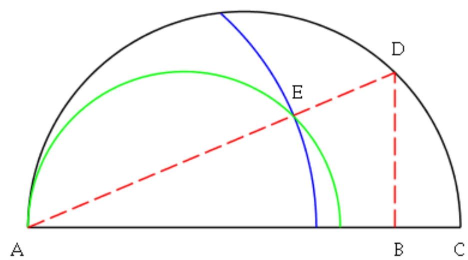

Figure 2: A failed attempted construction of an *Archytas configuration*

length. One such arc is shown in blue in Figure 2. Through each point E on that arc, there is exactly one semicircle tangent at A to the semicircle with diameter AC, shown in green in the figure. The line through A, E meets the outer semicircle at D and B is the foot of the perpendicular from D to AC. However, note that with this choice of E, DB *does not meet the smaller semicircle at all.* However, by rotating the segment AE about A and increasing the angle $\angle CAE$, we would eventually find that the corresponding BD cut through the corresponding smaller semicircle. Hence there must be some point E on the blue arc that yields an Archytas configuration as in Figure 1, by continuity. As we have described it, a naive process of finding that point might involve motion and exactly the sort of resort to "eyeballing" or use of the senses that Tyndares says Plato criticized in our passage from Plutarch![15]

[15]We can also easily locate such a point using modern coordinate geometry, trigonometry, and numerical root finding. But needless to say, all of that is well beyond the scope of Greek mathematics.

What Archytas actually did here has been interpreted in a number of different ways by different modern readers. One tradition[16] interprets Archytas' solution as a bold foray into solid geometry whereby a suitable point like our E in Figure 1 is found by the intersection of three different surfaces in three dimensions (a cylinder, a cone and the surface of revolution generated by rotating the semicircle with diameter AC about the tangent line at A). Very recently, a new interpretation[17] has been presented whereby the Archytas configuration is found by rotating one copy of the semicircle around A in the plane while keeping the other one fixed. In either our simple presentation, or the kinematic description of the three surfaces in three dimensions, or the new reading of Archytas' construction from [M], there is definitely an aspect of motion that seems to agree with Plato's reported characterization of the construction as μηχανικός. How the adjective ὀργανικός might apply is not clear, though.

The approach by Menaechmus is even more problematic. This can be described (very anachronistically) as follows.[18] Given line segments of lengths a, b, finding the two mean proportionals in continued proportion means finding x, y to satisfy:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}.$$

Hence, cross-multiplying and interpreting the resulting equations via coordinate geometry, we see the solution will come from the point of intersection of the parabola $ay = x^2$ and the hyperbola $xy = ab$. For this reason, Menaechmus has often been credited with initiating the study of the *conic sections*, later taken up and elaborated by Euclid, Archimedes, Apollonius, and other

[16] See for instance [H], pp. 246–249.

[17] See [M].

[18] This is essentially the presentation given in [H], pp. 252–255, although Heath also explains how the Greeks would have understood this via proportions.

Greek mathematicians.[19]

One of the mysterious aspects of the Platonic criticism recounted by Tyndares in our Plutarch passage is how the adjectives μηχανικός or ὀργανικός might apply to what Menaechmus did here. While it is true that the conic sections apart from the circle cannot be constructed as whole curves using only the Euclidean tools–the compass and straightedge–we do not know from any direct evidence that Plato was thinking of restricting the acceptable ways of making geometric constructions in precisely that way.[20]

In a final, decidedly odd, aspect of this story, Eutocius also gives a construction of the two mean proportionals in continued proportion that he ascribes to Plato himself. But that is by far the most mechanical and tool-based of all the solutions he describes in that it requires the use of a frame something like two "t-squares" joined along one edge. The configuration containing the two mean proportionals is found by maneuvering the device

[19]It must be said that the evidence that Menaechmus did any more than to stimulate those later developments on conic sections with his work on the Delian problem is somewhat flimsy. None of his own writings have survived and the discussion of his work in Eutocius uses the terminology for conic sections introduced much after the time of Menaechmus himself by Apollonius of Perga (262–190 BCE). We may surmise that either Eutocius himself or a source he consulted reworked Menaechmus' presentation in the light of later developments. Another piece of evidence is the epigram of Eratosthenes on the duplication of the cube that concludes the letter to Ptolemy III mentioned above. This includes an injunction not to " ... cut the cones in the triads of Menaechmus ... " to obtain a solution ([H], p. 246).

[20]Nevertheless, Steele argues in [S] that Plato's criticism of Eudoxus, Archytas, and Menaechmus recounted in this passage from Plutarch was a major contributing factor to the later, mistaken, notion that in Greek geometric constructions, *only* the compass and straightedge were acceptable tools. In fact Pappus of Alexandria (ca. 290–ca. 350 CE) gives two parallel statements of a classification of construction problems into three types depending on what sorts of auxiliary curves beyond the lines and circles that can be constructed with the Euclidean tools are required in Books III and IV of his Συναγωγή. Whether or not the duplication of the cube, the quadrature of the circle, and the trisection of a general angle could be accomplished using only the Euclidean tools remained an open question until the work of P. Wantzel and others in the 19th century CE. It is now known that *none* of these constructions is possible under those restrictive conditions. Many undergraduate mathematics majors learn proofs of these facts in abstract algebra courses.

until one point is found to coincide with one endpoint of one of the given segments.[21] A sort of νεῦσις, or limiting process, requring input from the senses of the geometer is the crucial component. As Knorr says, "one is astounded at the flexibility of the traditions which on the one hand attribute such a mechanism to Plato, yet on the other hand portray him as the defender of the purity of geometry and the sharp critic of his colleagues for their use of mechanical procedures in geometric studies."[22]

In conclusion, Plutarch has seemingly preserved a largely accurate picture of Plato's thinking, certainly more accurate than some of the traditions preserved in Eutocius' commentary. But from the work of Archytas and Menaechmus and from the later work of Archimedes, Apollonius and others I would argue that if something like Plato's criticism of the geometers in his circle actually happened at this point in history, then its effect on Greek mathematics was rather minimal. An openness to mechanical techniques was entering the mainstream of the subject in the Hellenistic and later periods.[23] While it drew on philosophy for its norms of logical rigor, mathematics had in essence emerged as an independent subject in its own right by the time of Eudoxus, Archytas, and Menaechmus. Plato's ideas about what should be its methods or goals were not the final word in a community of scholars caught up in the midst of an eminently successful ongoing research enterprise.

[21]There is a good diagram of this on page 58 of [K].

[22][K], p. 59

[23]A celebrated example of this is Archimedes' *Method of Mechanical Theorems*, which gives a systematic procedure, based on mechanics, to discover geometric area and volume formulas. Archimedes uses dissection procedures akin to the subdivisions used in modern integral calculus, combined with an idealized balance beam. Modern mathematicians are even happy to consider his arguments as complete proofs, although Archimedes himself had scruples about that point. However, this is certainly not the only such example. For instance, Heron of Alexandria (ca. 10 – ca. 70 CE) gives another different solution of the problem of the two mean proportionals in his Βελοποιίκα, a treatise on the design of artillery engines(!)

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