Plan for PURE Math 2012 Seminar

Week 3

Monday: Distance coordinates and configuration spaces
Tuesday: Central configurations in the 3- and n-body problems
Wednesday:
Thursday:

PURE Math 2012 - Seminar
Week 3 Labs - Understanding central configurations in the 3-body problem

Background

The Newtonian n-body problem is the study of the equations of motion of n point masses m_i at positions $q_i \in \mathbf{R}^d$, acted on by mutual gravitational forces following Newton's inverse square law of gravitation. The usual "$F = ma$" equation for the ith point mass reduces to the differential equation

$$\frac{d^2 q_i}{dt^2} = A_i = \sum_{j \neq i} \frac{m_j (q_j - q_i)}{r_{ij}^3}$$

where $r_{ik} = \|q_i - q_k\|$ is the mutual distance between the locations of the ith and kth masses. (The notation A_i is used since this is just the acceleration vector for the ith mass.) As we are seeing in class and in the discussions this week, *central configurations* in the Newtonian n-body problem are special configurations of masses that satisfy the algebraic equations

$$A_i + \lambda(q_i - c) = 0 \tag{1}$$

for some scalar λ, where $c = \frac{\sum_i m_i q_i}{\sum_i m_i}$ is the center of mass of the system.

If masses in a central configuration are released from rest, the acceleration is proportional to the vector from the initial position to the center of mass, and the whole configuration will collapse to a total collision at the center of mass (in which all the masses will reach the collision point at the same time). The intermediate positions will all be scaled ("homothetic") versions of the original configuration.

If the masses in a central configuration all lie in one plane, and they are released with suitable initial velocities, they can also move in periodic orbits.

We usually want to think of two central configurations as *equivalent* if there is a rigid motion of $\mathbf{R}^d$ that carries one into the other (together with a possible relabeling of the masses). This convention suggests that we should try to recast the equations (1) entirely in terms of the mutual distances r_{ij}, since they are invariants under the group of rigid motions of $\mathbf{R}^d$. As we have seen in class, the (asymmetric) Albouy-Chenciner ("AC") equations give one way to do this. A collection of mutual distances (r_{ij}) for $1 \leq i < j \leq n$ along with masses m_i defines a central configuration if and only if the following equations are satisfied for real $r_{ij} > 0$. We have effectively picked a particular distance scale for which $\lambda = 1$ in the following form of the AC equations. The AC system consists of the equations

$$G_{ij} = \sum_{k=1}^{n} m_k S_{ki} (r_{jk}^2 - r_{ik}^2 - r_{ij}^2) = 0, \tag{2}$$

where

$$S_{ki} = \begin{cases} r_{ki}^{-3} - 1 & \text{if } k \neq i \\ 0 & \text{if } k = i \end{cases}$$

(Note: the mutual distances are symmetric in the indices, so $r_{\ell m} = r_{m\ell}$ for all ℓ, m. This means that the S_{ki} with $k \neq i$ have the same symmetry: $S_{ki} = S_{ik}$.) We have one nontrivial equation for each pair $1 \leq i, j \leq n$ with $i \neq j$ (everything cancels out if $i = j$).

In the labs this week, we want to understand the solutions of the equations (2) in the case $n = 3$ for all choices of m_1, m_2, m_3 real and positive.

Historical Note: This determination of the central configurations in the 3-body problem goes all the way back to work of Euler and Lagrange in late 18th century, but they approached the problem in other ways! Interestingly enough, Lagrange's work on this problem contains one of the first occurences of the method of Lagrange multipliers that you studied in multivariable calculus.

Step 1 – Entering the Equations

Unfortunately, there is no really good way to have Sage deal with doubly subscripted variables directly. However we can have variables named `r12, r13, r23, S12, S13, S23, m1, m2, m3`, etc.

- Set up a "big" polynomial ring R with all of the above as variables, and a graded reverse lex order with the variables ordered as above.
- Define polynomials `G12, G13, G23` using the AC formulas in (2)
- Note that `S12, S13, S23` depend *rationally* on the `r12, r13, r23`. We want to get a system of polynomial equations out of this, so what we want is to substitute in the expressions for the `S12, S13, S23`, then take the *numerators* of the resulting rational functions (do you see why?) To do this simply in Sage, we can just apply the `.substitute()` and `.numerator()` methods in sequence. For instance:

  ```
  G12S = G12.substitute(S12 = 1/r12^3-1, S13 = 1/r13^3-1).numerator()
  ```

  will produce the correct polynomial we want from G_{12}.

Step 2 – Studying the AC Ideal

Continuing from the above,

- Make an ideal using `G12S` from above, and similar polynomials `G13S` and `G23S` produced in corresponding and appropriate fashion.
- Now compute a Gröbner basis for your ideal and examine *factored forms* the Gröbner basis polynomials one by one. You should see some very interesting factorizations like this in our output (possibly with the factors ordered differently):

  ```
  m3*m2*(r23-1)*r12*(r23^2+r23+1)*(r12-r13-r23)*(r12+r13-r23)\\
                    *(r12-r13+r23)*(r12+r13+r23)
  ```

(This is all one polynomial, multiplied together.) Now remember that we are looking for real, positive solutions of the AC equations in the case that the masses $m_i > 0$ for all i.

Step 3 – Breakdown into cases

At this point (already!), we are ready to break our problem down into several cases. The Gröbner basis we computed also generates the AC ideal, so any solution we are looking for must also be a solution of all of the equations obtained by setting the Gröbner basis polynomials equal to zero.

Q1. A number of the factors in the special factored polynomial above *cannot* be zero at the solutions we are interested in. Which ones are those? Only four of the factors here can actually be zero at the solutions we are interested in. Which are those?

Q2. Using Sage, show that

$$(r_{12} - r_{13} - r_{23})(r_{12} + r_{13} - r_{23})(r_{12} - r_{13} + r_{23})(r_{12} + r_{13} + r_{23})$$

is (possibly up to a sign) the Cayley-Menger determinant associated to the three points:

$$\Delta_{CM} = \det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 \end{pmatrix}$$

Q3. What is the geometric meaning, for instance, of the equation $r_{12} - r_{13} - r_{23} = 0$? (Recall, these variables represent the *mutual distances* between pairs of the point masses.)

For each of the factors that can be zero here, take that factor, adjoin it to the other generators of the AC ideal, and consider those new cases.

Step 4 – Analyzing the cases separately – Case I

For the case $r_{23} - 1 = 0$, a good strategy will be to

- Compute a Gröbner basis obtained by adjoining that polynomial to the AC ideal,
- Look for any new Gröbner basis polynomials that factor in "interesting" ways
- Try adjoining those factors to the Gröbner basis, and recompute a Gröbner basis from the resulting sets of polynomials.

What are all of the central configurations that fall into this case? What are they geometrically? How do the masses figure in? (For instance, is there any restriction on the m_1, m_2, m_3?) Also, are there solutions of the AC equations in this cases that do not correspond to central configurations? Explain. (Recall the r_{ij} and the masses m_i must be real and positive.)

Step 5 – The other case(s)

Q4. First, exactly how many other cases do we need to consider (at least if we are allowed to label the locations of the masses any way we want)? Is it one case or more than one?

Let's try an alternative strategy. Adjoin an appropriate factor from the factored form of the special polynomial (or the Cayley-Menger determinant found above), then force those polynomials into a new ring with a *lex* order designed to eliminate r_{12} and r_{13} and give an element of the ideal that is a univariate polynomial in r_{23} whose coefficients are polynomials in the masses m_1, m_2, m_3.

Q5. Compute the *lex* Gröbner basis and find the univariate polynomial in r_{23}. What is its degree in r_{23} (after removing any factors depending only on the masses and any factor that is a pure power of r_{23})? Do you notice anything special about the remaining polynomial after those nonvanishing factors are removed? There a way, for instance, to simplify the form (lower the degree), while keeping the number of real roots the same. What is that?

Q6. Try substituting at least 4 different combinations of rational values for the masses and finding the approximate roots of your simplified polynomial from Q4. (Force the substituted univariate polynomial into a single-variable ring, and use the method `.roots(ring=CC)` to find numerical approximations to all of the complex roots (Note: `CC` is Sage's floating point complex number system). How many real roots do you find in each case? Where are they located?

From the evidence generated in Q6, you will probably have come up with a conjecture about what is going on. But can we *prove something* about the number of real solutions of the univariate polynomial? There is a general method for answering this sort of question called *Sturm sequences*. Here is an outline of the way it works. Given a polynomial f, the *Sturm sequence* of a square-free polynomial f is computed by a process similar to the Euclidean algorithm for computation of $\gcd(f, f')$, *except that the negative of the remainder* is used rather than the remainder itself:

$$\begin{aligned} f_0 &= f \\ f_1 &= f' \\ f_i &= -\mathrm{rem}(f_{i-2}, f_{i-1}), i = 2, \ldots, m \end{aligned}$$

When $f_m \neq 0$ but $-\mathrm{rem}(f_{m-1}, f_m) = 0$ for the first time, we stop. The polynomial sequence $[f_0, f_1, \ldots, f_m]$ is called the Sturm sequence.

Then to determine the number of roots of f in any interval (a, b), you compute $s(a) - s(b)$, where $s(a)$ is the number of sign changes in the sequence of numbers

$$[f_0(a), f_1(a), \ldots, f_m(a)]$$

and similarly for $s(b)$. In counting sign changes, any zero terms in the sequence are ignored. Here is a simple example. Say $f_0 = x^4 - 3x^2 + x - 2$, then computing as above, check using Sage that the Sturm sequence is

$$[f_0, f_1, f_2, f_3, f_4] = \left[x^4 - 3x^2 + x - 2, 4x^3 - 6x + 1, \frac{3}{2}x^2 - \frac{3}{4}x + 2, \frac{31}{3}x + \frac{5}{3}, -\frac{8303}{3844}\right].$$

On the interval $[a, b] = [0, 2]$, note that $f(x) = 0$ has at least one real solution by the intermediate value theorem: $f(0) = -2 < 0$, while $f(2) = 4 > 0$. Is there more than one? Well the sequence of values at $x = 0$ is

$$\left[-2, 1, 2, \frac{5}{3}, \frac{-8303}{3844}\right]$$

which has $s(0) = 2$ sign changes. The sequence of values at $x = 2$ is

$$\left[4, 21, \frac{13}{2}, \frac{67}{3}, \frac{-8303}{3844}\right]$$

which has $s(2) = 1$ sign change. This gives $s(0) - s(2) = 1$, so Sturm's theorem says that there is exactly one real root of $f(x) = 0$ on the interval $[0, 2]$. Note that all of the above calculations are *exact* using rational arithmetic. This is the virtue of Sturm sequences – they are a purely symbolic computational method yielding exact results (no use of numerical approximations).

Q7. (This question is purposely very open-ended. Think of it as a first, smaller-scale research project.) Develop as much of a proof as you can to show that the univariate polynomial of degree 5 in $x = r_{23}^3$ obtained above in Q5 has exactly one real root in the interval $[0, 1]$ for all combinations of masses $m_1, m_2, m_3 > 0$ and no other real roots. Suggestions: It is good to think of using a unit of mass defined so that the smallest mass is 1, and the others have the form $1 + a$, $1 + a + b$ for some real $a, b \geq 0$. But then of course you will need to consider the locations of the different masses. Which is located where? How many different cases are there that you need to consider? Compute the Sturm sequence for the general degree 5 univariate polynomial in x, leaving a, b in the coefficients as symbolic constants. You can then get a feeling for signs in the Sturm sequence, for instance, by doing some 3D plotting of the Sturm sequence terms evaluated at $r_{23} = 0$ and $r_{23} = 1$ for $a, b \geq 0$. Can you prove your assertions completely?

Q8. Put your results together: What are all of the central configurations in the Newtonian 3-body problem?