Plan for PURE Math 2012 Seminar

Week 3

Monday and Tuesday: The Kepler problem
Wednesday and Thursday: Background on AC

PURE Math 2012 - Seminar

Week 3 Discussions - Background on Celestial Mechanics, Central Configurations

Days 1 and 2: The Kepler Problem

Background

Today we want start to retrace the steps of Isaac Newton (but using modern calculus notation!) to show how Kepler's Laws of planetary motion follow from the inverse square law of gravitation. This will involve mostly calculus and the geometry and algebra of vectors, rather than algebraic geometry. But *every* (future) mathematician should see this chain of reasoning (at least) once because it is truly a landmark in the application of mathematics to our understanding of the natural world!

The setting is this: place the large star mass M at the origin. We will assume that this is fixed – the mass of the orbiting planet is so much smaller than the star's mass that the planet's gravitational force on the star will be ignored. We will will use *polar* coordinates since they are better adapted to the geometry and represent the position of the moving mass m (the planet) by

$$q = r\mathbf{u}_r,$$

where

$$\mathbf{u}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$

is the unit vector in the direction of q. We will also use

$$\mathbf{u}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$

which is another unit vector making a right angle with $\mathbf{u}_r$.

We want to think of (r, θ), the polar coordinates of the moving planet, as functions of time, t throughout the following. Hence $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are also functions of t.

Discussion Questions

A) Show that

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta \text{ and } \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r$$

and hence by the chain rule

$$\frac{d\mathbf{u}_r}{dt} = \mathbf{u}_\theta\frac{d\theta}{dt} \text{ and } \frac{d\mathbf{u}_\theta}{dt} = -\mathbf{u}_r\frac{d\theta}{dt}.$$

B) Use the formulas from part A to show that the *velocity* and *acceleration* of the moving planet are as follows:

$$v = \frac{dq}{dt} = r\frac{d\theta}{dt}\mathbf{u}_\theta + \frac{dr}{dt}\mathbf{u}_r$$

and

$$a = \frac{d^2q}{dt^2} = \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right) \mathbf{u}_\theta + \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right) \mathbf{u}_r$$

The formulas from part B and the $F = ma$ law give the *equations of motion* in this coordinate system: If the force acting on the planet is written as $F = F_r\mathbf{u}_r + F_\theta\mathbf{u}_\theta$, then we have

(1)
$$m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right) = F_\theta$$

and

(2)
$$m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right) = F_r$$

C) The force exerted by the star on the planet is a *central force*, which means that $F_\theta = 0$. Show that this implies

(3)
$$r^2\frac{d\theta}{dt} = h$$

is constant. Hint: Multiply through by r in (1) after setting the right hand side equal to zero.

D) In calculus, you may have seen that the area inside a parametric polar curve $r(t) = f(\theta(t))$ for $t \in [t_1, t_2]$ is given by

$$A = \int_{t_1}^{t_2} r(t)^2\frac{d\theta(t)}{dt}\ dt$$

(If not, can you see why this is true? Think of chopping the region into a large number of "wedges" by lines through the origin, and approximating each wedge by a circular sector.) Deduce Kepler's Second Law (equal areas swept out in equal times) from part C.

E) Now assume that the central force exerted by the star on the planet has the form given in Newton's gravitional law:

$$F_r = \frac{-km}{r^2}$$

(where $k = GM$), so (using (3))

(4)
$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = \frac{-k}{r^2}$$

F) We want to solve this differential equation. Show that if we let $w = \frac{1}{r} + \frac{k}{h^2}$ (and express r as a function of θ), then (4) is equivalent to

$$\frac{d^2w}{d\theta^2} + w = 0$$

which has general solution

$$w = A\cos\theta + B\sin\theta$$

for constant A, B.

G) By rotating the coordinate system, we can take $A = 0$ and $B > 0$. Show that then

$$r = \frac{h/k^2}{1 + e\cos\theta} \tag{5}$$

where $e = Bh^2/k > 0$. Show also that (5) is the equation of an ellipse with one focus at the origin (the constant c is the *eccentricity* of the ellipse). This is Kepler's First Law(!)

H) Note that the minimum distance to the star (like (perihelion for the Earth) occurs for $\theta = 0$ and the maximum distance (like aphelion) occurs for $\theta = \pi$. Use this observation show that the semimajor axis is

$$a = \frac{1}{2}\left(\frac{h^2/k}{1+e} + \frac{h^2/k}{1-e}\right) = \frac{h^2}{k(1-e^2)}$$

The semimajor axis, semiminor axis, and eccentricity of an ellipse are related by $b^2 = a^2(1-e)$. Deduce that

$$b^2 = \frac{h^2a}{k}.$$

I) The area is the ellipse is πab. From part D, deduce that the period of the orbit of the planet (its "year") is $T = \frac{2\pi ab}{h}$. Then use this and part H to deduce Kepler's Third Law (T^2 is proportional to a^3).

Days 3 and 4: Mutual Distances, Cayley-Menger, Albouy-Chenciner

Background

We have now seen the way the $\binom{n}{2}$ mutual distances r_{ij} can be used to describe configurations of points in a Euclidean space, and the Albouy-Chenciner form of the central configuration equations.

Discussion Questions

A) Show that the Cayley-Menger determinant for a configuration of $n = 3$ points is a constant multiple of the square of the area of the corresponding triangle if the points are not collinear. Also, what happens if the points *are* collinear? Hint: Heron's Formula. Look this up if you have not seen it before.

B) What is the corresponding statement for the Cayley-Menger determinant of $n = 4$ points? Prove your assertion.

C) Work through the derivation of the symmetrized Albouy-Chenciner equations from pages 3 and 4 of the Hampton-Moeckel article "Finiteness of Relative Equilibria of the Four-Body Problem" and write up the argument in your own words, filling in necessary details.

D) How does the *asymmetric* form of the Albouy-Chenciner equations as we have defined them follow from this?