PURE Math 2012 – Ideas for Research Projects
First Draft – April 28, 2012

The following are my initial ideas for the research projects the students at PUREMath 2012 will work on. I am expecting to have 4 groups of 3 students each formed from the *Residents* at the program working on these for the last 4.5 - 5 weeks of the program. There are another 2 or 3 ideas I have in various states of development at this point.

Schedule and Logistics

[Fill in later: How will groups be chosen? How will topics be assigned, or will the groups get to chose? Who will be the primary contact people for which groups? What will the daily schedule during the research phase look like? What will deadlines for various components be? What will be the expectations for progress reports, final technical reports, presentations, etc.]

A good starting point for context and ideas – all projects

[S] Saari, Donald G. Central Congurations–A Problem for the Twenty-first Century, `math.uci.edu/ dsaari/BAMA-pap.pdf`

Idea 1 – The "Planar Restricted 4-body Problem"

Comments: Getting really good numerically-computed results here is a nice computational challenge, although much of this has been done before in different forms and quite a bit has been claimed on the basis of numerical evidence. Formal proofs of any sort would be even better, though they are even harder to come by in this problem.

Although the central configurations in the 3-body problem have been essentially *completely* known for many years, the same cannot really be said even now for the 4-body problem. There are a number of open problems concerning the possible central configurations with various specific collections of masses, or with particular geometries. For this project, you would investigate one variation that has attracted some interest in work of J. Kulevich, G. Roberts and C. Smith ([KRS]), and J. Barros and E. Leandro ([BL]). If three of the bodies have much, much larger mass than the fourth (think of the system consisting of an inhabited planet with two typical moons, and a much, much smaller satellite launched from the planet), then we can essentially treat the fourth mass as *zero* and look at possible central configurations in this situation. If all four bodies have center of mass in a single plane, this is called the "planar restricted 4-body problem" in the sources below.

If the three large masses are normalized (e.g. by setting one of them to 1, or by making the sum equal to 1), then it is an interesting problem to try to determine exactly how many central configurations there are for each combination of masses, where the changeovers or *bifurcations* occur, and exactly what occurs at the bifurcations.

For this project the goals would be to

(a) First, get a feeling for what is known in this area by looking at the references below.

(b) Do some experimentation in `Sage` to get the "lay of the land." How many solutions do there seem to be for different combinations of the large masses? What if two of them are equal?

(c) Are there actually just three different possiblities for the numbers of central configurations? Suppose the three large masses are normalized to $(1, m_2, m_3)$. In the first quadrant of the (m_2, m_3) plane, which regions correspond to which numbers?

References

[BL] Barros, Jean F.; Leandro, Eduardo S. G. The set of degenerate central configurations in the planar restricted four-body problem. *SIAM J. Math. Anal.* **43** (2011), no. 2, 634–661.

[KRS] Kulevich, Julianne L.; Roberts, Gareth E.; Smith, Christopher J. Finiteness in the planar restricted four-body problem. *Qual. Theory Dyn. Syst.* **8** (2009), no. 2, 357–370.

[L] Leandro, Eduardo S. G. On the central configurations of the planar restricted four-body problem. *J. Differential Equations* **226** (2006), no. 1, 323351.

Idea 2 – Finiteness Questions with Negative Masses

Comment: This is probably the most open-ended of these topics – it's an area where I have looked at a lot of things that *have not* worked to produce examples of the kind described below. Nevertheless

Perhaps the major unsolved problem about central configurations is whether the number of such configurations for a given collection of masses $m_i > 0$ is always finite. The answer is known to be yes for the $n = 3, 4$-body problems, but this is not quite known for $n = 5$, and essentially completely open in general for $n \geq 6$. One cautionary example is given by the article [R] below, which shows that in the planar 5-body problem, if one of the masses is allowed to be negative, then the finiteness fails – there is an infinite (one real parameter) family of central configurations in that case for the same collection of masses. This example seems quite "generalizable," but to my knowledge, no one has found any additional examples of such families (allowing some negative masses). It would be quite interesting to know whether there are any!

The goals of this project would be to

(a) Start by verifying Roberts' example using the Albouy-Chenciner equations (that's not the approach he used), then

(b) Explore other related configurations to see whether similar phenomena occur with $n > 5$ masses (some allowed to be negative. "Anything goes" here since you are essentially looking for a new example that no one has seen before. But some ideas would be to look at cases where you have more than one "ring" of masses about a central mass, look at other cases where the configuration can "flex" to an extent leaving some of the distances fixed, etc.

(c) On the other hand, might Roberts' example be the *only one* of this type? That seems unlikely to me, but it's certainly possible. If you could understand how to make the phrase "of this type" more precise and prove that, it would be a major result.

References

[R] Roberts, G.E., A continuum of relative equilibria in the ve-body problem, *Physica D*, **127** (1999), 141–145.

Idea 3 – Central Configurations of Vortices

The Albouy-Chenciner equations for central configurations of masses under the Newtonian gravitation law can be generalized in many ways. One that also has real physical interest concerns the problem of central configurations for any number $n \geq 3$ of "Helmholtz vortices." Consider a perfect fluid with an infinite horizontal surface and a constant thickness. Some states of the fluid where small whirlpools are present are well described by giving the positions and the vorticities (a measure of "strength" of the rotation) of a finite number of "point vortices." (Some famous experiments by A. M. Mayer simulate Helmholtz vortices by a device where N identical magnets are oating on a surface of water. Equations similar to those developed by Helmholtz are quite frequent in physical models.) There exist relative equilibria, i.e. configurations analogous to planar central configurations in the Newtonian problem that remain unchanged up to rotation and translation. For instance, collinear configuration of N vortices exist for all N.

In algebraic terms, the equations describing configurations of vortices have the same form as the Albouy-Chenciner equations, but in which the $S_{ij} = \frac{1}{r_{ij}^3} - 1$ are replaced by the somewhat simpler $S_{ij} = \frac{1}{r_{ij}^2} - 1$. The m_i must also be interpreted as the vorticities, which can can be positive or negative real numbers.

For this project, the main goals would be to

(a) Work through the 3-vortex problem using our algebraic tools and `Sage` and understand the similarities and differences with the Newtonian problem.

(b) Investigate some special cases of the 4-vortex problem. For instance, to what extent do the results of Cors and Roberts (see [CR] below) about co-circular configurations extend to the vortex case? Are there parallel statements about the possible geometries of vortex configurations?

References

[AF] Albouy, A.; Fu, Y. Euler configurations and quasi-polynomial systems. *Regul. Chaotic Dyn.* **12** (2007), no. 1, 39–55.

[CR] Cors, Josep; Roberts, G. Four-Body Co-Circular Central Configurations, *Nonlinearity*, **25**, no. 2, 343-370.

Idea 4 – Real Root Counting and Central Configurations

Comment: I think this is probably the most theoretically (algebraically) demanding project idea in this list, and hence probably suitable only for a really strong group, although there might also be enough to do here to split up the aspects and have two groups work in partial collaboration.

The recent work of M. Hampton and R. Moeckel (see [HM] in references) on planar central configurations (relative equilibria) in the 4-body problem applied a theoretical "big gun" from contemporary algebraic/polyhedral geometry *"BKK theory"*. What they showed using this was that the Albouy-Chenciner equations for the 4-body problem have only finitely many *complex* solutions with $r_{ij} \neq 0$ for all $1 \leq i, j \leq 4$, for all positive collections masses m_i. M. Hampton and A. Jensen have continued this approach for the 5-body problem as well, although the results to date are less definitive.

Even though this technique can be effective for proving finiteness, the real interest in celestial mechanics is not in the collection of all complex solutions of these equations. Instead, what we want ideally is a way to count the number of *real, positive solutions* (recall that the r_{ij} represent mutual distances in the physical interpretation of the system) and then determine each of them to any desired accuracy. Now the "BKK theory" applies to give an upper bound on the number of positive real solutions, of course, since $\mathbf{R}_{>\mathbf{0}} \subset \mathbf{C}$. But there are typically many more nonreal solutions than real solutions. In addition, some real solutions may contain values $r_{ij} \leq 0$, so those are not physically relevant either. Finally, it would be very desirable to count real solutions with $r_{ij} > 0$ by *symbolic, exact methods* that do not rely on sensitive numerical root-finding.

One possibility that has not really been explored yet in any of the work in celestial mechanics is to apply some fairly well-developed techniques for real root-counting that can be seen as multivariate analogs of the Sturm systems that we applied in the labs in week 3. If we can compute a Gröbner basis (with respect to any monomial order) for a *zero-dimensional* ideal I – and that is an unfortunately big "if" for some of the systems of interest here – then there are symbolic techniques based on linear algebra in the quotient ring $A = \mathbf{Q}[x_1, \ldots, x_s]/I$ that can be used to determine the number of real solutions in any region defined by polynomial inequalities (such as $x_i > 0$ for all i).

The main goals of this project topic would be:

(a) to learn about these symbolic real root-counting methods,
(b) to work out exactly how they could be applied in general to count the number of positive, real solutions of a system of polynomial equations, code them for use in `Sage`, and finally
(c) to experiment on some cases of interest in celestial mechanics (starting with the 3-body problem from the labs in week 3) and (if feasible) going to some more complicated cases as well.

References

[CLO2] Cox, D., Little, J., and O'Shea, D. Using Algebraic Geometry, 2nd ed., Springer 2005 (especially Chapter 2 and references there).

[HM] Hampton, M. and Moeckel, R. Finiteness of relative equilibria of the four-body problem. *Invent. Math.* **163** (2006), no. 2, 289–312.