Singularities of Offset Curves and Surfaces

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Abstract

In this paper we discuss several aspects of the relationship between offsets and singularities of algebraic curves. In particular we prove that any curve will always be an irreducible component of its secondary offset. We use this result to elaborate on a result by Alcazar and Sendra as well as to describe which combinations of singularities can occur on an offset. We also analyze the effect of offsetting on an isolated singularity of a surface in special cases.

1 Introduction

1.1 *Offset Curves*

In a similar manner to Alcazar and Sendra in [1], we can informally describe an offset of a curve $\mathcal{C}$ as a curve that is parallel to $\mathcal{C}$ at a given distance d. However, we require a more precise definition to carry out our work. We require that our curve $\mathcal{C}$ be a real irreducible algebraic curve and we let R be the set of all regular points z in $\mathcal{C}$ (that is, the set of all nonsingular points of $\mathcal{C}$). If $\nabla\vec{f}(z) \neq 0$, we can discuss $\frac{\nabla\vec{f}}{\|\nabla\vec{f}\|}$, the unit normal vector of the curve $\mathcal{C}$, and use the expression $z \pm d\frac{\nabla\vec{f}}{\|\nabla\vec{f}\|}$ to find two points a distance d from the curve at z. If we take the union of these points we get two sets

$$A_d(\mathcal{C}) = \left\{ z + d\frac{\nabla\vec{f}}{\left\|\nabla\vec{f}\right\|} : z \in R \right\}$$

and

$$A_{-d}(\mathcal{C}) = \left\{ z - d\frac{\nabla \vec{f}}{\left\|\nabla \vec{f}\right\|} : z \in R \right\};$$

This leads to our first definition below:

Definition 1 (Offset of a Curve)**.** *We call the offset the smallest algebraic variety containing both $A_d(\mathcal{C})$ and $A_{-d}(\mathcal{C})$, that is, the Zariski closure $\mathcal{O}_d(\mathcal{C}) = \overline{A_d(\mathcal{C}) \cup A_{-d}(\mathcal{C})}$.*

In much of our work we use parameterizations of a curve $\mathcal{C}$ and we employ different notation. We work with the parameterization $p(t) = (x(t), y(t))$ and denote the offset curve by

$$p_1(t) = p(t) \pm \frac{d}{\delta(t)^{\frac{1}{2}}}(-y'(t), x'(t)),$$

where $\delta(t) := x'(t)^2 + y'(t)^2$. Clearly, this formula is equivalent to the formula $f(z) = z \pm d\frac{\nabla \vec{f}}{\|\nabla \vec{f}\|}$. We will also employ notation

$$p_1(t) = p(t) \pm d\frac{p'(t)^{\perp}}{\|p'(t)\|}, \tag{1}$$

where $p'(t)^{\perp}$ denotes the vector $(-y'(t), x'(t))$ perpendicular to the vector $p'(t)$.

1.2 *Places of a Curve*

In order to determine what offsets do to singularities, we must establish some definitions that allow us to formally describe the local shape of a curve. We accomplish this using two concepts known as places of a curve and the signature of a place. Using the definition from Alcazar and Sendra [1], we call $\mathcal{P}(t)$ a place of $\mathcal{C}$ when $\mathcal{P}(t)$ is an equivalence class of irreducible (all powers in the expansions do **not** have a common factor greater than 1) local parameterizations of $\mathcal{C}$ around a point P. In order to derive a place that gives us meaningful information we first consider the local parameterization of curves given by the formal power series representations

$$\begin{aligned} x(t) &= \alpha_0 + \alpha_{r_1} t^{r_1} + \alpha_{r_2} t^{r_2} + \dots, \\ y(t) &= \beta_0 + \beta_{s_1} t^{s_1} + \beta_{s_2} t^{s_2} + \dots, \end{aligned}$$

where $f(x(t), y(t)) = 0$, $0 < r_1 < r_2 < \cdots$ and $0 < s_1 < s_2 < \cdots$. In this form there are many different but equivalent parameterizations.

We define the signature of a place below following [1]:

Definition 2. *Let $\mathcal{P}(t)$ be a real place of $\mathcal{C}$. The signature of $\mathcal{P}(h)$ is defined as the pair (p, q) where p is the first non-zero natural number such that $\mathcal{P}^{(p)}(0) \neq (0, 0)$, and $q > p$ is the first natural number such that $\mathcal{P}^{(p)}(0), \mathcal{P}^{(q)}(0)$ are linearly independent.*

In order to obtain a standard form, we simplify the expression by making a change of variables (see Walker [2])

$$t = c_1\bar{t} + c_2\bar{t}^{\,2} + c_3\bar{t}^{\,3} + \ldots$$

where $c_1 \neq 0$. After a change of parameter of this form, a translation, and if necessary an orhtogonal change of coordinates in $\mathbb{R}^2$, every place can be brought into the form

$$x(\bar{t}) = \alpha_p \bar{t}^{\,p} \qquad (2)$$

$$y(\bar{t}) = \beta_q \bar{t}^{\,q} + \beta_{q+1}\bar{t}^{\,q+1} + \ldots, \qquad (3)$$

where $p < q$ are the components of the signature.

Using the definition of signature, it is apparent that the leading terms in the local parameterizations dominate the local behavior of the curve. In particular, it is the powers of these leading terms that provide the most information about the singularity.

1.3 *Our Problem*

One of our main goals is to understand what the offsetting process does to the local shape of a curve. In particular, we determine what happens to singularities like cusps and elaborate on the results given by Alcazar and Sendra in their paper "Local shape of offsets to algebraic curves". The difficulty in the explanation results because cusps on the generator do not always generate cusps on the offset.

Another of our goals is to figure out what types of singularities an offset curve can have. In a simple case, we determine if it is possible to construct a curve with two singularities at different points. We also address the question of whether it is possible to construct a curve with any number of singularities at any chosen points in the 2-dimensional plane.

Finally, one last goal is to work in $\mathbb{R}^3$ with an offset surface. We consider the surface $z^n = h(x, y)$ where $h(x, y)$ is an homogeneous polynomial in x, y and seek to understand what the offsetting process does to the local shape of a surface of this form.

1.4 *Results from Alcazar and Sendra*

In their paper, Alcazar and Sendra give a nearly complete description of how the offsetting process affects the local shape of the curve. They accomplish this by describing what happens to the signature of the place in the offsetting process. We will outline their method below.

We begin by considering a place as in (2) and (3),

$$\mathcal{P}(t) = (x(t), y(t)) = (\alpha_p t^p, \beta_q t^q + \beta_{q+1} t^{q+1} + \ldots).$$

and use the equation (1) given above for the offset curve,

$$\mathcal{P}_1(t) = p(t) \pm \frac{d}{\sqrt{x'(t)^2 + y'(t)^2}}(-y'(t), x'(t)).$$

Because we know the power series for $x(t)$ and $y(t)$ it is easy to find the series for $x'(t)$, $y'(t)$, and $\phi(t) = x'(t)^2 + y'(t)^2 = a_0 + a_r t^r + O(t^{r+1})$ for some $r \geq 1$. With this information, we can calculate the first few (lowest power) terms of the new place by making use of the identity

$$\frac{1}{\sqrt{\phi(t)}} = \frac{1}{\sqrt{a_0}} - \frac{a_r}{2a_0^{3/2}} t^r + O(t^{r+1}).$$

Finally, after making the appropriate substitutions we have the formula that $\mathcal{P}_1(t)$ equals

$$\left(\alpha_p t^p \pm d\left(\frac{q\beta_q}{p\alpha_p} t^{q-p} + \ldots\right), \beta_q t^q \pm dp\alpha_p\left(\frac{1}{p\alpha_p} - \frac{q^2\beta_q^2}{2p^3\alpha_p^3} t^{2(q-p)} + \ldots\right)\right).$$

To find the new signature (p^*, q^*) we must simply find the lowest power of t in each coordinate. We have four cases for the new signature. It is clear that some special behavior happens when $q = 2p$ and $d = d_{crit} = \frac{\alpha_p^2}{2\beta_q}$. Following Alcanzar and Sendra, we will call this the critical offsetting distance, d_{crit}.

Theorem 1 ([1], Theorem 7)**.** *Let (p, q) be the signature of a real place of the generator curve $\mathcal{C}$. Then the signature (p^*, q^*) of the corresponding place of the offset curve $\mathcal{O}_d(\mathcal{C})$ is as follows.*

1. *If $q - 2p > 0$, then $(p^*, q^*) = (p, q)$.*

2. *If $q - 2p = 0$ and $d = d_{crit} = \frac{\alpha_p^2}{2\beta_q}$, then $p^* > p$ and $q^* > q$.*

3. *If* $q-2p=0$ *and* $d \neq d_{crit}$*, then* $(p^*, q^*) = (p, q)$.

4. *If* $q-2p<0$*, then* $(p^*, q^*) = (q-p, 2q-2p)$.

Proof. In case 1, $p < q-p$ and $q < 2(q-p)$ so p and q are the still lowest exponents in the formula given above and the signature will not change. In case 4 on the other hand, $q-p < p$ and $2(q-p) < q$ so these smaller exponents are the new lowest powers and signature. If $q-2p=0$ and $d \neq d_{crit}$ as in case 3, then no cancellation occurs and p and q are still the lowest exponents and the new signiture will equal the old as in case 1. In case 2, when $q-2p=0$ and $d = d_{crit}$, the lowest power terms shown in the formula above cancel with eachother and we must look to higher power terms to find the new lowest exponent. This gives the inequalities $p^* > p$ and $q^* > q$ claimed above. $\square$

Note that this characterization is incomplete as the description of what happens in case 2 is somewhat unsatisfactory. There are no more obvious conclusions about (p^*, q^*) that can be made from the argument above. We therefore seek to improve the description in this case by other means. It is this problem that we address below.

2 Offset at the Critical Distance

Our first goal is to investigate what happens in case 2 from Theorem 1. Before we can answer this question we need some results about offsets and places on a curve.

Lemma 1. *If* $p(t)$ *and* $p_1(t)$ *are regular points of* $\mathcal{C}$ *and* $\mathcal{O}_d(\mathcal{C})$ *respectively, then* $p'(t)$ *and* $p_1'(t)$ *are parallel vectors.*

Proof. Given a local parameterization $p(t) = (x(t), y(t))$ of an algebraic curve $\mathcal{C}$, we can parameterize the offset curve $\mathcal{O}_d(\mathcal{C})$ locally using the equation

$$p_1(t) = p(t) + \frac{d}{\delta(t)^{\frac{1}{2}}}(-y'(t), x'(t))$$

and the notational convenience $\delta(t) := x'(t)^2 + y'(t)^2$.
From this expression we can see that

$$p_1' = p' + \frac{d}{\delta^{\frac{3}{2}}}\Big(-\delta y'' + y'(x'x'' + y'y''), \delta x'' - x'(x'x'' + y'y'')\Big),$$

which equals

$$p' + \frac{d}{\delta^{\frac{3}{2}}}(-y''((x')^2+(y')^2)+y'x'x''+(y')^2y'', x''((x')^2+(y')^2)-(x')^2x''-x'y'y'')$$

$$= p' + \frac{d}{\delta^{\frac{3}{2}}}(x'y'x''-(x')^2y'', (y')^2x''-y'y''x') = \Big[1+\frac{d}{\delta^{\frac{3}{2}}}(y'x''-y''x')\Big]p'.$$

□

Lemma 2. *Let $\mathcal{C}$ be an irreducible algebraic curve. Then $\mathcal{C}$ is an irreducible component of $\mathcal{O}_d(\mathcal{O}_d(\mathcal{C}))$. Moreover, if z is a singular point of $\mathcal{C}$ and $\mathcal{P}$ is a place of $\mathcal{C}$ centered at z, then $\mathcal{P}$ is also a place of $\mathcal{O}_d(\mathcal{O}_d(\mathcal{C}))$ centered at z.*

Proof. All but finitely many points of $\mathcal{C}$ and $\mathcal{O}_d(\mathcal{O}_d(\mathcal{C}))$ are regular points. Take a local parametrization $p(t)$ for $\mathcal{C}$ such that $z = p(0)$ is a regular point of $\mathcal{C}$. Then by (1), a local parametrization of $\mathcal{O}_d(\mathcal{C})$ near the point corresponding to z with offset distance $+d$ is

$$p_1(t) = p(t) + d\frac{p'(t)^{\perp}}{\|p'(t)\|}.$$

For all but finitely many t, $p_1(t)$ will be a regular point of $\mathcal{O}_d(\mathcal{C})$. We then calculate a parametrization $p_2(t)$ of the branch of the secondary offset $\mathcal{O}_d(\mathcal{O}_d(\mathcal{C}))$ with offset distance $-d$:

$$\begin{aligned} p_2(t) &= p_1(t) - d\frac{p_1'(t)^{\perp}}{\|p_1'(t)\|} \\ &= p(t) + d\frac{p'(t)^{\perp}}{\|p'(t)\|} - d\frac{p_1'(t)^{\perp}}{\|p_1'(t)\|}. \end{aligned}$$

But since $p'(t)$ is parallel to $p_1'(t)$ by Lemma 1, we get simply $p_2(t) = p(t)$ whenever $p(t)$ and $p_1(t)$ are regular points on the corresponding curves. Thus one branch of the secondary offset has infinitely many points in common with the original curve. It follows from Bézout's Theorem (see IVA [3]) that $\mathcal{C}$ is a component of the secondary offset.

The second part of the statement of the lemma follows from the definition of the offset of a singular curve as the Zariski closure of the union of the interior and exterior offset curves on the regular points as in [1] (Definition 1). If z is a singular point of $\mathcal{C}$ and $\mathcal{P}$ is a place of $\mathcal{C}$ centered at z, then we can choose a sequence $\{t_n\}$ such that $\mathcal{P}(t_n)$ is regular for all n and $\mathcal{P}(t_n) \to z$ as $n \to \infty$ to compute the point z' on $\mathcal{O}_d(\mathcal{C})$ with offset distance $+d$. The argument above shows that whenever the corresponding point $\mathcal{P}_1(t_n)$ on the offset is also regular, then the corresponding point $\mathcal{P}_2(t_n)$ on the secondary offset for the distance $-d$ is the same as $\mathcal{P}(t_n)$. By passing to a subsequence if necessary, we conclude that $\mathcal{P}$ is also a place of $\mathcal{O}_d(\mathcal{O}_d(\mathcal{C}))$ centered at z. $\square$

Next we return to the result given in [1]. Using the notation

$$(p, q) := sign(\mathcal{P}) = sign(\mathcal{P}'')$$

$$(p^*, q^*) := sign(\mathcal{P}')$$

recall that the analysis divided into four separate cases:

1. $q - 2p > 0$
2. $q - 2p = 0$ and $d = d_{crit}$ (cancellation occurs in the expression for $\mathcal{P}'$)
3. $q - 2p = 0$ and $d \neq d_{crit}$
4. $q - 2p < 0$,

where the signature stays the same for cases 1 and 3, decreases only in case 4 and increases only in case 4. Now that we have the result about secondary offsets, we have the tools to elaborate on what happens in case 2.

Theorem 2. *Every pair (p^*, q^*) with $q^* - 2p^* < 0$, $p = q^* - p^*$, and $q = 2q^* - 2p^*$ occurs as the signature of a place on an offset curve of an algebraic generator curve whose corresponding place has signature (p, q). Conversely, an offset of a place with signature (p, q), $q = 2p$, and $d = d_{crit}$ must yield a place with signature of this type.*

Proof. Using secondary offsets, we make deductions about what happens in case 2 by using our information about the other cases and interchanging which of the curves we think of as the generator and which as the offset. Consider what happens to the signature as we compute each offset and pass from $\mathcal{P}$ to $\mathcal{P}'$ to $\mathcal{P}''$. If the first transition falls under case 2 then we know that $p^* > p$ and $q^* > q$, so we know that $p^* > p^{**}$ and $q^* > q^{**}$. Since the coordinates of the signature must now decrease, the previous discussion tells us that the second transition must be of the type in case 4. That is $q^* - 2p^* < 0$, $p = p^{**} = q^* - p^*$, and $q = q^{**} = 2q^* - 2p^*$. This information gives us some restrictions on the possible values of $sign(\mathcal{P}')$ in case 2. In fact, it is not hard to see that it is possible get any pair (p^*, q^*) satisfying these three conditions. Simply construct a curve with a place of signature (p^*, q^*) (case 4) and take the offset. Then the corresponding place on the offset will have signature (p, q) (case 2) and its offset (really a secondary offset and so the original place) will have the desired signature. Combining the above conclusions with the original characterization also leads to the observation that $q - p$ is invariant under the transition from $\mathcal{P}$ to $\mathcal{P}'$. $\square$

Corollary 1. *If $\mathcal{P}'$ is the place on $\mathcal{O}_d(\mathcal{C})$ corresponding to $\mathcal{P}$ of $\mathcal{C}$, $sign(\mathcal{P}) = (p, q)$, and $sign(\mathcal{P}') = (p^*, q^*)$ then*

$$q^* - p^* = q - p.$$

3 Combinations of Singular Points

Our second goal is to give a general description of what combinations of singularities can occur on an offset. Since every curve is an offset of any of its own offsets, making any curve with any singularities will prove that offset curves can have all types of singularities in arbitrarily large quantities. We construct the curve by giving explicit equations for a parameterization $x(t), y(t)$. Since every polynomial parameterization can be implicitized, our construction must yield an algebraic curve. Further, since the image of the parameterization will be contained in one irreducible component (see [3], Chapter 4, Section 5), we may claim without loss of generality that the curve we construct is irreducible.

Theorem 3. *Let (p_i, q_i), $i = 1, ..., n$ be any collection of signatures, then we can construct an algebraic curve with singularities of the same types.*

Proof. We construct the parameterization in three steps.

Step 1. Let

$$x(t) = \prod_{i=1}^{n}(t - i)^{p_i} \qquad y(t) = \prod_{i=1}^{n}(t - i)^{q_i}.$$

Here we have n places of interest, one corresponding to each of the values $i = 1, ..., n$. To find the parameterization centered at the jth place, we simply make the substitution $t \to t + j$. From this substitution and the form of the original parameterization, it is easy to see that the lowest power of t in the expression for x will be p_j, while the lowest power of t in the expression for y will be q_j. This seems to be exactly what we want. However, since we have no guarantee that these parameterizations will be irreducible. We change our initial formula to ensure that this does not occur.

Step 2. Consider

$$\begin{aligned} x(t) &= \prod_{i=1}^{n}(t-i)^{p_i} \\ y(t) &= \prod_{i=1}^{n}\Big[(t-i)^{q_i} + c_i(t-i)^{q_i+1}\Big]. \end{aligned}$$

Here the c_i are nonzero constants in $\mathbb{R}$. Note that the added terms do not change the lowest powers of t in the parameterizations corresponding to the places $i = 1, \ldots, n$ unless $1 + c_i(t-i)$ is divisible by some $t-j$. This only occurs when $j = i - \frac{1}{c_i}$. To prevent this we demand that the inequalities $c_i \neq \frac{1}{i-j}$ hold for all $i \neq j$. Note that this only prevents us from considering finitely many n-tuples $(c_1, \ldots, c_n)$. To ensure that the place corresponding to $i = j$ is irreducible, it suffices to arrange for the coefficient (call it k_j) of t^{q_j+1} in the expansion for y (after the substitution $t \to t + j$) to be nonzero. Each k_i is a polynomial in the n variables $c_1, \ldots, c_n$. k_j can be calculated directly from the original parameterization. Fortunately, it is unnecessary to do so completely. We only need to find the coefficient of c_j in k_j. Using the original parameterization, we compute that this coefficient is

$$\prod_{i \neq j}(j-i)^{q_i}.$$

Since $c_i \neq \frac{1}{i-j}$, this number is nonzero. We conclude that k_j cannot be the zero polynomial. Hence its zero set $\mathbb{V}(k_j)$ has measure zero in $\mathbb{R}^n_{c_1,\ldots,c_n}$. Let $\mathbf{B}_0$ be the set of all $(c_1, \ldots, c_n) \in \mathbb{R}^n$ such that $c_i = \frac{1}{i-j}$ for some pair (i, j) and let $\mathbf{B} = \mathbf{B}_0 \cup \bigcup_{i=1}^n \mathbb{V}(k_i)$. Then $\mathbf{B}$ has measure 0 in $\mathbb{R}^n$. In particular, $\mathbf{B} \neq \mathbb{R}^n$ and $\exists\, \mathbf{c} = (c_1, \ldots, c_n)$ such that $\mathbf{c} \notin \mathbf{B}$. We choose this $\mathbf{c}$ to give the values of our coefficients. The lowest powers of t are still the same as in part 1 because $\mathbf{c} \notin \mathbf{B}_0$. Furthermore, the parameterizations at the places we are interested in will be irreducible since $\mathbf{c} \notin \mathbb{V}(k_j)$ guarantees that $k_j(\mathbf{c}) \neq 0$. The only problem remaining is that all of the places we have constructed will be centered at the same point. The next (and last) modification will fix this by moving the centers of the places to arbitrary points (x_i, y_i).

Step 3. Now consider

$$x(t) \;=\; \prod_{i=1}^{n}(t-i)^{p_i} + \sum_{h=B}^{M_x} \alpha_h t^h$$

$$y(t) \;=\; \prod_{i=1}^{n}\Big[(t-i)^{q_i}+c_i(t-i)^{q_i+1}\Big]+\sum_{h=B}^{M_y}\beta_h t^h,$$

where $M_x, M_y, B \in \mathbb{Z}$ and $\alpha_h, \beta_h \in \mathbb{R}$ are constants to be determined. The goal in adding these terms is to change the center of the places without having any effect on either their signature or their irreducibility. This means that we want to add constant without adding any other low power terms, to obtain

$$\sum_{h=B}^{M_x}\alpha_h(t+j)^h = x_j + O(t^{p_j+1}) \qquad \sum_{h=B}^{M_y}\beta_h(t+j)^h = y_j + O(t^{q_j+2}).$$

The equations corresponding to x and α are of exactly the same type as those corresponding to y and β. To show that we can always construct sums with this property, we show that given arbitrary x_j, $M>0$ and $B>M$, there exist constants a_i such that

$$\sum_{h=B+1}^{B+nM} a_h(t+j)^h = x_j + O(t^M),$$

for all $1 \le j \le n$. This is equivalent to the system of linear equations

$$\sum_{h=B+1}^{B+nM} a_h j^h = x_j, \qquad \sum_{h=B+1}^{B+nM} a_h j^{h-l}\binom{h}{l} = 0 \qquad 1 \le j \le n,\ 1 \le l \le M-1.$$

We rewrite this system as

$$\sum_{h=B+1}^{B+nM} a_h j^{h-B} = x_j j^{-B} \qquad \sum_{h=B+1}^{B+nM} a_h j^{h-B}\binom{h}{l} = 0.$$

Using linear combinations transforms this to

$$\sum_{h=B+1}^{B+nM} a_h j^{h-B} = x_j j^{-B} \qquad \sum_{h=B+1}^{B+nM} a_h j^{h-B} h^l = 0,$$

and finally

$$\sum_{h=B+1}^{B+nM} a_h j^{h-B} = x_j j^{-B} \qquad \sum_{h=B+1}^{B+nM} a_h j^{h-B}(h-B)^l = k_{jl},$$

or

$$\sum_{u=1}^{nM} a_h j^u = x_j j^{-B} \qquad \sum_{u=1}^{nM} a_h j^u u^l = k_{jl},$$

which has coefficient matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 2 & 2 & 2 & \dots & 3 & 3 & \dots \\ 1 & 2 & 4 & \dots & 2^2 & 2^2\cdot 2 & 2^2\cdot 4 & \dots & 3^2 & 3^2\cdot 2 & \dots \\ 1 & 3 & 9 & \dots & 2^3 & 2^3\cdot 3 & 2^3\cdot 9 & \dots & 3^3 & 3^3\cdot 3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Each row corresponds to a value of h, each column corresponds to a value of l, and each block of columns corresponds to a value of j. We will show that the system can be solved by proving that this matrix is invertible by demonstrating that its columns are linearly independent. Note that the columns can be represented by the functions

$$\begin{pmatrix} 1 & i & i^2 & \dots & 2^i & 2^i i & 2^i i^2 & \dots & 3^i & 3^i i & \dots \end{pmatrix}.$$

Since submatrices obtained by omitting blocks of columns at the right end will have the same form, it suffices by induction to show that each column vector in the rightmost block of the matrix is linearly independent of those to its left. This is achieved by constructing a linear functional ϕ whose kernel contains every column to the left of this vector, but not the vector itself. Before constructing ϕ, there are two families of linear maps that we need to introduce. We define the family of discrete derivatives $\partial : \mathbb{R}^r \to \mathbb{R}^{r-1}$ for $r \geq 2 \geq$ and the exponent maps $E_\gamma : \mathbb{R}^r \to \mathbb{R}^r$ respectively by

$$\partial : \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_r \end{pmatrix} \longmapsto \begin{pmatrix} v_2 - v_1 \\ v_3 - v_2 \\ \dots \\ v_r - v_{r-1} \end{pmatrix} \qquad E_\gamma : \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_r \end{pmatrix} \longmapsto \begin{pmatrix} \gamma v_1 \\ \gamma^2 v_2 \\ \dots \\ \gamma^r v_r \end{pmatrix}.$$

Lemma 3. *The discrete derivative ∂ satisfies:*

1. *For all $i < j$, $x^i \in \ker(\partial^j)$.*

2. *$x^j \notin \ker(\partial^j)$.*

3. *$\partial[x^j n^x] = cx^j n^x + p(x)n^x$, where $\deg(p) < j$ and $c = 0$ if and only if $n = 1$.*

Since the discrete derivative reduces the degree of any polynomial by one but does not destroy (or increase) the highest power of the product of a polynomial with an exponential, we are led to the following construction: The functional corresponding to the qth vector in the rightmost block will be

$$\phi_q = \partial^q \circ E_{\frac{n-1}{n}} \circ \partial^{M^*+1} \circ \ldots \circ E_{\frac{2}{3}} \circ \partial^{M^*+1} \circ E_{\frac{1}{2}} \circ \partial^{M^*+1}.$$

By the properties listed above, ϕ_q has the desired kernel, and the construction is finished. $\square$

4 Offsets of Singular Surfaces

Finally we want to describe what offsetting does to singularities on surfaces. While the surface case is significantly more complicated, a similar formula exists for calculation the parameterization of an offset and we were able to analyze a family of special cases. Specifically, we want to investigate what happens at the offset at the origin of the surface

$$z^n = h(x, y),$$

where $\deg(h) = m < n$ and h is a homogeneous polynomial (a polynomial of the form $\sum_{i=0}^{m} c_i x^i y^{m-i}$). We further restrict the analysis by assuming that this is an isolated singularity. This prevents the partial derivatives $\partial_x h$ and $\partial_y h$ from vanishing simultaneously at any point except the origin. The surface can be parametrized and a normal vector can be computed at all smooth points as follows:

$$x(u,t) = u \quad y(u,t) = t \quad z(u,t) = h(u,t)^{\frac{1}{n}}$$

$$\partial_u = \left(1, 0, \frac{1}{n} h(u,t)^{\frac{1-n}{n}} \frac{\partial h}{\partial u}(u,t)\right)$$

$$\partial_t = \left(0, 1, \frac{1}{n} h(u,t)^{\frac{1-n}{n}} \frac{\partial h}{\partial t}(u,t)\right)$$

$$\vec{n} = \partial_u \times \partial_t = \left(-\frac{1}{n} h(u,t)^{\frac{1-n}{n}} \frac{\partial h}{\partial u}(u,t), -\frac{1}{n} h(u,t)^{\frac{1-n}{n}} \frac{\partial h}{\partial t}(u,t), 1\right).$$

Given a parameterization of a curve intersecting the origin by

$$u(\gamma) = c_\alpha \gamma^\alpha + c_{\alpha+1} \gamma^{\alpha+1} + \ldots$$

$$t(\gamma) = k_\beta \gamma^\beta + k_{\beta+1}\gamma^{\beta+1} + \dots,$$

where without loss of generality $\beta > \alpha$, we calculate $\vec{n} =$

$$(-\frac{1}{n}h(u(\gamma),t(\gamma))^{\frac{1-n}{n}}\frac{\partial h}{\partial u}(u(\gamma),t(\gamma)), -\frac{1}{n}h(u(\gamma),t(\gamma))^{\frac{1-n}{n}}\frac{\partial h}{\partial t}(u(\gamma),t(\gamma)), 1).$$

Theorem 4. *The set of limiting normals* $\lim\limits_{\gamma\to 0} \frac{\vec{n}}{\|\vec{n}\|}$*, considering different limiting curves, will always be an arc in the unit circle in the xy-plane.*

Proof. Here we can calculate a power series in γ for each coordinate so in fact it don't matter what limiting curve we use. Our interest will be in the lowest power terms since they dominate the behavior in the limit.

If A is the lowest power of γ in h, B is the lowest power in $\frac{\partial h}{\partial u}$, and C is the lowest power in $\frac{\partial h}{\partial t}$, then from the above equation we can see that the lowest powers of γ in x and y in the expression for $\vec{n}(\gamma)$ will be given by $\frac{1-n}{n}A + B$ and $\frac{1-n}{n}A + C$ respectively.

We have two cases. If the lowest power of $h(u(\gamma), t(\gamma))$ is $\gamma^{m\alpha}$ and comes from a term of the form u^m then $A = m\alpha$. Further, the smallest term of $\frac{\partial h}{\partial u}$ will be of the form u^{m-1} so $B = (m-1)\alpha$. Thus

$$\deg_\gamma(x) = \frac{1-n}{n}A + B = \frac{1-n}{n}m\alpha + (m-1)\alpha$$
$$= \left(\frac{1}{n} - 1\right)m\alpha + m\alpha - \alpha = \left(\frac{m}{n} - 1\right)\alpha < 0.$$

If on the other hand, the lowest power of γ in $h(u(\gamma), t(\gamma))$ is $\gamma^{a\alpha+b\beta}$ and comes from a term of the form $u^a t^b$ with $b \neq 0$ then $A = a\alpha + b\beta$. The lowest term of $\frac{\partial h}{\partial t}$ will come from a term of the form $u^a t^{b-1}$ so $C = a\alpha + (b-1)\beta$. Then

$$deg_\gamma(y) = \frac{1-n}{n}A + C = \frac{1-n}{n}(a\alpha + b\beta)) + a\alpha + (b-1)\beta$$
$$= \left(\frac{1}{n} - 1\right)(a\alpha + b\beta) - (a\alpha + b\beta) - \beta = \frac{1}{n}(a\alpha + b\beta) - \beta \leq \frac{1}{n}(m\beta) - \beta$$
$$= \left(\frac{m}{n} - 1\right)\beta < 0.$$

In either case, one of the coordinates has a negative power of γ as its dominant term and must diverge as $\gamma \to 0$. The limit of the norm will diverge as well, so the limit of the normalization will have zero z-coordinate and be a vector in the unit circle in the xy-plane. Considering different limiting curves will give different points on the unit circle so that the limit points of the offset corresponding to the origin on the generator will give an arc of the unit circle. To determine which arc, we have to calculate the limit of the normalization of the vector $(x(\gamma), y(\gamma))$ given by $\vec{n}(\gamma)$. This means calculating the limit

$$\lim_{\gamma \to 0} \left(\frac{\frac{\partial h}{\partial u}}{(\frac{\partial h}{\partial u}^2 + \frac{\partial h}{\partial t}^2)^{1/2}}, \frac{\frac{\partial h}{\partial t}}{(\frac{\partial h}{\partial u}^2 + \frac{\partial h}{\partial t}^2)^{1/2}} \right).$$

Since the numerators and denominators in this expression are homogeneous in γ to equal degrees, we can separate and cancel equal powers of γ from each so that $u(\gamma)$ and $t(\gamma)$ are effectively replaced by

$$u^*(\gamma) = c_\alpha + c_{\alpha+1}\gamma + \cdots$$

$$t^*(\gamma) = k_\beta \gamma^{\beta-\alpha} + k_{\beta+1}\gamma^{\beta-\alpha+1} + \cdots$$

Since the variables (u, t) no longer limit to the origin in the limit, and since both partial derivatives cannot vanish simultaneously anywhere else, the new expression for the limit is no longer an indeterminate form and can be evaluated directly at the point (c_α, k_α). This gives a parameterization of the arc generated by the singularity by

$$\begin{aligned} \phi : S^1 &\rightarrow S^1 \\ (x, y) &\mapsto \left(-\frac{\frac{\partial h}{\partial u}(x, y)}{(\frac{\partial h}{\partial u}(x, y)^2 + \frac{\partial h}{\partial t}(x, y)^2)^{1/2}}, -\frac{\frac{\partial h}{\partial t}(x, y)}{(\frac{\partial h}{\partial u}(x, y)^2 + \frac{\partial h}{\partial t}(x, y)^2)^{1/2}} \right). \end{aligned}$$

□

For example if the generator surface is $z^5 = x^3 + y^3$, ϕ is given by

$$(x, y) \mapsto \left(-\frac{3x^2}{3(x^4 + y^4)^{1/2}}, -\frac{3y^2}{3(x^4 + y^4)^{1/2}} \right)$$

and the arc generated for positive offset distance d will only be the quarter circle in the third quadrant.

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