Mathematics 304, section 2 – Ordinary Differential Equations
Selected Solutions – Problem Set 1
September 13, 2004

Section 1.2/2. The problem said to find a differential equation of the form

$$\frac{dy}{dt} = 2y - t + g(t) \tag{1}$$

that has $y = e^{2t}$ as a solution. To find the unknown part $g(y)$, substitute for y on both sides in (1):

$$2e^{2t} = 2e^{2t} - t + g(e^{2t})$$

and cancel to get $0 = -t + g(e^{2t})$, or $g(e^{2t}) = t$. Hence $g(y) = \frac{1}{2}\ln(y)$ will give one such equation, since $\frac{1}{2}\ln(e^{2t}) = t$.

Section 1.2/30. Solve:

$$\frac{dy}{dt} = \frac{t}{(1-t^2)y}; \quad y(0) = 4$$

Separating variables and integrating we have

$$\int y\ dy = \int \frac{t}{1-t^2}\ dt$$

The integral on the right has the form $-\frac{1}{2}\int \frac{du}{u}$ for $u = 1 - t^2$. Hence we get

$$\frac{1}{2}y^2 = \frac{-1}{2}\ln|1-t^2| + c$$

From the initial condition, $c = 8$, so

$$y^2 = 16 - \ln|1-t^2| \Rightarrow y = \sqrt{16 - \ln|1-t^2|}$$

(positive square root, since $y(0) > 0$).

Section 1.2/35. Let S be the amount of salt at time t. Then the information given in the problem says that S is a solution of

$$\frac{dS}{dt} = \frac{1}{4} - \frac{1}{10}S; \quad S(0) = 0$$

(the initial condition comes from the fact that at the start we have "pure water" – no salt at the start). This is a separable equation with solution

$$S(t) = \frac{5}{2} - \frac{5}{2}e^{\frac{-t}{10}},$$

so

$$
\begin{aligned}
S(1) &\doteq .238 \\
S(10) &\doteq 1.58 \\
S(60) &\doteq 2.49 \\
S(1000) &\doteq 2.50 \\
\lim_{t\to+\infty} S(t) &= 2.50
\end{aligned}
$$

Note that this result makes sense because "in the long run" the water in the bucket should have the same salt concentration as the salt plus water mixture entering.

Section 1.2/37. The differential equation describing Newton's Law of Cooling is

$$\frac{dT}{dt} = k(T - 70)$$

where T is the temperature of the cup of cocoa, and k is the constant of proportionality. Solving this by separating variables and integrating, we get

$$T(t) = 70 + ce^{kt}$$

The initial condition is the initial temperature: $T(0) = 170$, so

$$T(t) = 70 + 100e^{kt}.$$

We also know that the temperature is decreasing by 20 degrees per minute at $t = 0$, so $T'(0) = -20$. This information determines the value of k: $-20 = 100ke^{k\cdot 0}$, so $k = -1/5$. Then to find the time when the temperature reaches 110 degrees,

$$110 = 70 + 100e^{-t/5} \Rightarrow t = -5\ln(.4) \doteq 4.58$$

(The cocoa has cooled down to 110 degrees after about 4.6 minutes.)

Section 1.2/40. The hardest part of this one is just getting at the essential points buried in the amusing story. This is like the other "mixing problems", except that after the initial mistake, *no more hot sauce is added to the chili*, so the in-flow rate term is actually *zero*. Keeping track of the units, if H is the amount of hot sauce in the remaining chili, in teaspoons, as a function of t,

$$\frac{dH}{dt} = \frac{-1}{32}H; \quad H(0) = 12$$

(just an exponential decay equation!) The solution is $H(t) = 12e^{-t/32}$. The time when the remaining hot sauce is 4 teaspoons (2 teaspoons per gallon in the 2 gallon pot of chili) is the solution of

$$4 = 12e^{-t/32} \Rightarrow t = -32\ln(1/3) \doteq 35$$

That is, the recipe level of hot sauce is regained after about 35 minutes, and about 35 cups of the chili have been served by that time.

Section 1.8/12. Solve

$$\frac{dy}{dt} = \frac{2t}{1+t^2}y + \frac{2}{1+t^2}; y(0) = -2.$$

This is a first order linear equation with

$$\begin{aligned} e^{\int g(t)\ dt} &= e^{\int \frac{2t}{1+t^2}\ dt} \\ &= e^{\ln(1+t^2)} \\ &= 1+t^2 \end{aligned}$$

Hence applying the general formula we get

$$\begin{aligned} y &= ce^{\int g(t)\ dt} + e^{\int g(t)\ dt}\int e^{-\int g(t)\ dt} r(t)\ dt \\ &= c(1+t^2) + (1+t^2)\int \frac{2}{(1+t^2)^2}\ dt \end{aligned} \tag{2}$$

Pulling the 2 out of the remaining integral, we need to compute $\int \frac{dt}{(1+t^2)^2}$ to finish. This is one where we need to use the method of *trigonometric substitution* (or a table of integrals. Here's the method if you do it "by hand": The form $1+t^2$ suggest the tangent substitution $t = \tan(\theta)$ so $dt = \sec^2(\theta)d\theta$. The reason for this substitution is the identity $1+\tan^2(\theta) = \sec^2(\theta)$. Hence we have

$$\begin{aligned} \int \frac{dt}{(1+t^2)^2} &= \int \frac{\sec^2(\theta)}{\sec^4(\theta)}\ d\theta \\ &= \int \cos^2(\theta)\ d\theta \\ &= \frac{1}{2}(\sin(\theta)\cos(\theta)+\theta) \end{aligned}$$

Then since $t = \tan(\theta)$, $\theta = \arctan(t)$, and $\sin(\theta) = \frac{t}{\sqrt{1+t^2}}, \cos(\theta) = \frac{1}{\sqrt{1+t^2}}$. Substituting these in, we find:

$$\int \frac{dt}{(1+t^2)^2} = \frac{1}{2}\left(\frac{t}{1+t^2} + \arctan(t)\right)$$

Substituting this back into (2), we get the general solution:

$$y = c(1+t^2) + t + (1+t^2)\arctan(t)$$

The initial condition $y(0) = -2$ implies $c = -2$, and we're done.