Mathematics 304, Section 2 – Ordinary Differential Equations
Solutions Midterm Exam 1 – October 1, 2004

Directions: Do all work in the blue exam booklet. You may use a calculator and the table of integrals at any point. There are 100 total regular points and 10 Extra Credit points.

I. Consider the first order ODE

$$y' = 2t(y-2)^{2/3} \tag{1}$$

A) (15) Find a solution of this equation that satisfies the initial condition $y(0) = 3$.

Solution: This is a separable equation, so we solve as usual:

$$\int (y-2)^{-2/3}\, dy = \int 2t\, dt$$
$$3(y-2)^{1/3} = t^2 + c$$
$$y = \left(\frac{t^2}{3} + c\right)^3 + 2$$

From the initial condition,

$$3 = c^3 + 2 \Rightarrow c = 1.$$

So the solution is

$$y = \left(\frac{t^2}{3} + 1\right)^3 + 2$$

B) (10) What does the Existence and Uniqueness Theorem say about solutions of the initial value problem for (1) with $y(0) = 2$? Show that the constant function $y(t) = 2$ for all t is one solution. Is it the only one?

Solution: The right-hand side function in this equation is

$$f(t, y) = 2t(y-2)^{2/3} \Rightarrow \frac{\partial f}{\partial y} = \frac{4t}{3(y-2)^{1/3}}$$

The partial derivative function is not continuous on any rectangle in the (t, y)-plane containing points with $y = 2$. Hence, the hypotheses of the Existence and Uniqueness Theorem are *not satisfied*. However, the most we can say in this case is that we are not guaranteed that there exists a unique solution to the initial value problem with $y(0) = 2$. (There could be more than one, but that's not guaranteed either.) In this case there is more than one solution. First, $y = 2$ for all t satisfies the differential equation since

$$y' = 0 = 2t(2-2)^{2/3}$$

is true for all t. Hence this constant function is a solution. But then, we can also get a solution of this ODE with $y(0) = 2$ by using the solutions found in part A. If $y(0) = 2$, we get $c = 0$ and a different solution

$$y = \left(\frac{t^2}{3}\right)^3 + 2 = \frac{t^6}{27} + 2.$$

II. (25) A tank with capacity 200 gal. originally contains 50 gal. water with 30 lb of salt in solution. Water containing .5 lb of salt per gallon is entering at 3 gal./minute and the tank is thoroughly mixed so the salt concentration is uniform at all times. Contents of the tank are also flowing out of an exit pipe at 2 gal/minute. Find the amount of salt in the tank at all times prior to the time the tank overflows. What is the amount of salt at the instant the tank starts to overflow?

Solution: From the statement of the problem, letting $S =$ the total amount of salt in the tank (not salt per gallon, the total), we have $50 + t$ gallons of liquid after t minutes, and

$$S' = 1.5 - \frac{2}{50 + t} S$$

This is a first order linear equation with $g(t) = \frac{-2}{50+t}$ and $r(t) = 1.5$. Solving as usual,

$$e^{\int g(t)\ dt} = e^{-2\int \frac{dt}{50+t}} = e^{-2\ln|50+t|} = (50 + t)^{-2}$$

Then

$$y(t) = c(50 + t)^{-2} + \frac{3}{2}(50 + t)^{-2} \int (50 + t)^2\ dt$$

$$= c(50 + t)^{-2} + \frac{1}{2}(50 + t)^{-2}(50 + t)^3$$

$$= c(50 + t)^{-2} + \frac{1}{2}(50 + t)$$

The initial condition is $S(0) = 30$ (total salt in tank at $t = 0$). Then

$$\frac{c}{2500} + 25 = 30 \Rightarrow c = 12500$$

Finally, since the capacity of the tank is 200 gallons, the tank starts to overflow when $50 + t = 200$ or $t = 150$ minutes. At that time, the total amount of salt is

$$S(150) = \frac{12500}{200^2} + \frac{1}{2}(200) = 100.31$$

pounds of salt.

III. Let T, M be positive constants with $T < M$, and consider the autonomous 1st order ODE

$$P' = -P\left(1 - \frac{P}{T}\right)\left(1 - \frac{P}{M}\right) \tag{2}$$

A) (15) What are the equilibrium solutions of this equation? Which of the equilibria are sinks and which are sources?

Solution: Setting

$$-P\left(1 - \frac{P}{T}\right)\left(1 - \frac{P}{M}\right) = 0$$

, we see $P = 0$, $P = T$, or $P = M$. Since this is a cubic polynomial in P with a negative leading coefficient and three horizontal axis intercepts, $P' > 0$ if $P < 0$, or $T < P < M$. $P' < 0$ if $0 < P < T$ or $P > M$. Hence $P = 0$ and $P = M$ are sinks, while $P = T$ is a source.

B) (10) Suppose the equation (2) models growth of a population of a species of organisms in a particular environment. Based on your answer to part A, what happens to solutions of initial value problems for (1) if $P(0) < T$? if $T < P(0) < M$? if $P(0) > M$? What are the meanings of the P values $P = T$ and $P = M$?

Solution: If $0 < P(0) < T$, then the solution decreases and tends to $P = 0$ as $t \to +\infty$; if $T < P(0) < M$, then the solution increases and tends to $P = M$ as $t \to +\infty$; if $P(0) > M$, then the solution decreases and tends to $P = M$ as $t \to +\infty$. In biological terms, M represents the maximum sustainable population, while T represents the minimum sustainable population (or population threshold). If $P(0) < T$, then the population would eventually die out.

IV. (25) Sketch the bifurcation diagram for the family of ODE

$$y' = y^3 - 3ay$$

containing the parameter a. Describe all bifurcations that occur.

Partial Solution: This is a "pitchfork bifurcation". There are equilibria at $y = 0$ for all a, and at $y = \pm\sqrt{3a}$ for $a > 0$. For $a \leq 0$, $y = 0$ is a source. For $a \geq 0$, the "new" equilibria $y = \pm\sqrt{3a}$ are sources, while $y = 0$ changes to a sink.