Mathematics 304, section 2 – Ordinary Differential Equations
Lab Day 1 – Maple for Differential Equations, Bifurcations, etc.
September 20, 2004

Background

Today, we want see how to use Maple to plot direction fields and approximate solutions for first order ODE. We will also look at further examples of bifurcations in the family of equations

$$x' = ax - x^3 - b$$

where a, b are two independent real parameters.

Maple Commands for Plotting Functions, Direction Fields and Solutions

The Maple commands that we will use today are:

`plot` to plot ordinary functions of one variable
`animate` to see animated 2D plots
`DEplot` to plot direction fields and solutions of an ODE

The basic Maple command for 2D plotting graphs of the form $y = f(x)$ is called `plot`. The basic format is

```
plot(function,range,options);
```

where: `function` is the function to be plotted – the simplest way to specify one is via a formula, `range` is the range of x-values you want to see plotted, and `options` can be used to control the form of the plot if desired. No options need be specified, however, so that part can be absent. For example

```
plot(x^4-x^3+2*x-5*sin(x^2),x=1..3,title="My First Maple Graph");
```

will plot

$$y = x^4 - x^3 + 2x - 5\sin(x^2)$$

on the range $1 \le x \le 3$, and add a title.

An example of the `animate` command is given in the Lab Questions below.

The `DEplot` command is part of the `DEtools` package; to use it you will need to start by entering the command:

```
with(DEtools):
```

to load the `DEtools` routines. Note: the colon at the end of a Maple command *suppresses the output* – the command is executed but the results are not displayed. In this case the output would be a (long) list of the commands that are parts of the package – maybe interesting, but wasteful of paper when you go to print out your worksheet!

For instance, suppose we wanted to study the direction field, and solutions of various initial value problems for the 1st order ODE $y' = ty - 1$. Specifically, say we wanted to plot the solution of the initial value problem with $y(0) = 1$ for $t \in [-1, 1]$ (this is chosen arbitrarily here). We could use the following commands. First we define the slope function $f(t, y) = ty - 1$:

```
f := (t,y) -> t*y - 1;
```

Then we set up the differential equation in Maple's format:

```
eq := diff(y(t),t) = f(t,y(t));
```

Then we use `DEplot` to generate the plot we want:

```
DEplot(eq,y(t),t=-1..1,[[y(0)=1]],linecolor=black);
```

When you execute the `DEplot` command the output will be a plot showing the direction field (with line segments drawn as arrows pointing in the direction of increasing x), together with a plot of the solution.

Comments:

1. The 3-step process above could actually be combined into a single command if you put the definition of the differential equation directly into the `DEplot` command (instead of building it up using the slope function). I recommend that you do things this way though, at least at first, to keep everything straight in your mind and minimize the chance for typing errors.
2. The general format of the `DEplot` command is

   ```
   DEplot(equation,depvar(indepvar),range,inits,options);
   ```

   where `equation` is the differential equation, in the format given above. After the equation comes the name of the dependent variable (i.e. the unknown function) with the independent variable in parentheses, then the range of values of the independent variable you want to see plotted, then the initial conditions, and options last. *The order of these is fixed – you cannot mix them up freely.* You can plot several different solutions together by putting more than one initial condition inside the outer pair of square brackets, separated by commas, e.g. `[[y(0) = 1],[y(0)=2],[y(-1)=2]]`. Options can be used to control the way solutions are plotted and the appearance of the plot. The `linecolor=black` in the above is an option, for instance. Without that, Maple would use a default yellow color that does not show up when you print your worksheet(!)
3. To draw the solution, Maple is using an *approximate numerical method* similar to, but more powerful than, the *Euler's Method* you may have studied in calculus. Methods of this type produce a table of values for an approximate solution function by "following the direction field". Then Maple plots the approximate solution by connecting the points with straight line segments. *This method is far from fool-proof!* In particular,

if the slope function is changing rapidly, discontinuous at some points, etc., it can lead to inaccurate results. Fortunately, there are ways to try to fine-tune its behavior. If you notice that your approximate solution looks very jagged (adjacent straight line segments have very different slopes) or if it jumps around wildly, you can try including an option

```
stepsize=.01
```

or some number even smaller than .01. The step size is the spacing between successive values of the independent variable when Maple computes the approximate solution. Reducing the stepsize from its default value can improve plots. *It also slows down the computation, though, because more points must be computed!* If the slope function is discontinous at some point on your solution, this may not help; in some cases, this type of method will always fail no matter how small the step size is!

4. *IMPORTANT NOTE:* Some initial conditions with some differential equations lead to solutions that become unbounded extremely quickly. If this happens Maple will probably "punt" on trying to draw them and return a blank graph with a message that says: *Floating Point Overflow. Please shorten axes.* This means that you need to *reduce the range of values of the independent variable.*

Lab Questions

All parts of questions A-E in this lab refer to the family of first order ODE's

$$x' = ax - x^3 - b. \tag{1}$$

The goal here is to understand the bifurcations that occur in this family, and to generate a sort of "2D bifurcation diagram" illustrating the possibilities. Some of the parts of these questions ask you to generate hand sketches of phase lines, bifurcation diagrams, and so forth. You can do those on separate sheets and attach them to your Maple printout.

A) Set $a = 1$ in the general form (1)

1) With $b = -2$, then $b = 0$, then $b = 2$ use the `DEplot` command described above to plot approximate solutions of this equation with initial conditions $x(0) = -2, -1/2, 1/2, 2$ (all on the same axes). Sketch the phase line for this equation with $a = 1$ and each of these b-values (by hand) and explain how they relate to your solution graphs.
2) Plot $y = x - x^3 - b$ (ordinary function plot) for various b. For another way to see this, you may find it helpful to use animation:

```
with(plots):
animate(x-x^3-b,x=-2..2,b=-2..2);
```

The toolbar buttons that show up when you left click over the plot region act like the controls on a cassette tape player (rememeber those??) Use them to "play the animation". (Note: the default gives 16 "frames" in the animation – 16 equally spaced b-values starting at $b = -2$, increasing to $b = +2$.) Using the information from these plots, sketch the bifurcation diagram for the family $x' = x - x^3 - b$ as b varies (you'll want to do this by hand).

B) Sketch the bifurcation diagram for the family with $a = 0$ and b varying. You may want to repeat calculations as in question A to see the pattern. Include any Maple plots you generate in your lab write-up.

C) Sketch the bifurcation diagram for the family with $a = -1$, and b varying. Include any Maple plots you generate in your lab write-up.

D) Now consider the families with $b = -1, 0, 1$ and a varying. Sketch the bifurcation diagram for each of these families. Include any Maple plots you generate in your lab write-up.

E) ("Putting it all together") The goal now is to construct a plot of the a, b-plane showing the regions where $x' = ax - x^3 - b$ has different numbers of equilibrium points.

1) Show in general that if $f(x)$ is a polynomial of any degree and $f(x)$ has a multiple root (i.e. double or higher multiplicity) at $x = c$, then

$$f(c) = f'(c) = 0.$$

2) Show that for a cubic of the form $f(x) = ax - x^3 - b$, there exists some c with $f(c) = f'(c) = 0$ if and only if

$$4a^3 - 27b^2 = 0.$$

The expression $4a^3 - 27b^2$ is called the *discriminant* of the cubic – it's a generalization of the $b^2 - 4ac$ under the radical sign in the quadratic formula for the roots of $ax^2 + bx + c = 0$(!) If you take our Abstract Algebra course (MATH 351-352), you may see discriminants of polynomials again in more generality.

3) What are the different numbers of equlilibrium points in the family? Which regions in the (a, b)-plane correspond to ODE's $x' = ax - x^3 - b$ with each possible number?

4) Describe using phase lines and the graph $y = ax - x^3 - b$ the bifurcations that occur when you cross from one region into another. Is there a difference depending on whether the cross-over happens at $(0, 0)$ or a point different from $(0, 0)$?

F) Let's add on a periodic term like in our discussion of periodic harvesting in class. Consider the differential equation

$$x' = x - x^3 - (0.1)\sin(2\pi t)$$

Study the solutions and determine whether there appear to be any periodic solutions.

Assignment

Lab writeups due Friday, September 24. (This is the same day as the due date for Problem Set 3; the problem set is very short this time for this reason.)