Mathematics 304, section 1 – Ordinary Differential Equations
Discussion 1 – "More Mixed Up"
September 8, 2004

Background

Last week, recall that we considered an example involving cholesterol levels in the bloodstream and modeled that real world system by a (separable) first order ODE. Today, we will consider other similar examples involving the idea of "mixing". The basic patterns behind all of these examples are:

- we have a system in which we are tracking the amount of some quantity present,
- there one or more ways in which that quantity is being added and subtracted, so
- writing y for the amount as a function of t, we have: *the rate of change of y is equal to the (total) in-flow rate of y, minus the total out-flow rate of y*:

$$\frac{dy}{dt} = (\text{inflow rate}) - (\text{outflow rate}) \tag{1}$$

Discussion Questions

I. (Imagine that we are in the Heinz pickle factory preparing to produce a batch of Polish dills.) A large tank holds 1000 liters of pure water when a salt solution begins to flow in at the rate of 6 liters per minute. The salt concentration in the liquid entering is 1 kilogram/liter. The mixture in the tank is kept well-stirred at all times (so the salt concentration is *kept uniform throughout the tank*). At the same time, brine is being removed from the tank at the same rate: 6 liters per minute.

A) Let $y(t)$ denote the amount (i.e. mass) of the salt in the tank as a function of time. Explain why (1) leads to the ODE

$$\frac{dy}{dt} = 6 - \frac{3}{500}y.$$

B) When will the concentration of salt in the tank reach 1/2 kilogram/liter?
C) What happens to the concentration of salt in the tank if we let $t \to \infty$? Is your answer reasonable? Why or why not?

II. Now, suppose we have the same situation as in question I, but the out-flow rate is 5 liters per minute, rather than 6 liters per minute.
A) What changes in the differential equation? (Be careful; note what happens to the volume of the mixture in the tank as t increases.)
B) Solve your new differential equation and determine what happens to the concentration of salt in the tank if we let $t \to \infty$ again.
C) If the capacity of the tank is (only) 8000 liters, though, is our differential equation a good model? Why? How might we include this feature of the real-world system?

III. The equation

$$\frac{dy}{dt} = -2y + ty^{-2} \tag{2}$$

is not first order linear. (Why not?)

A) Show however, that the substitution $v = y^3$ (and some algebra) reduces the equation to the first order linear form:

$$\frac{dv}{dt} = -6v + 3t.$$

B) Solve this equation, then substitute $v = y^3$ back in to derive the general solution of (2). This is an example of a *Bernoulli equation*; Bernoulli equations can always be reduced to first order linear equations by a similar process.

Assignment

Group write-ups, due in class, Monday September 13.