Mathematics 304 – Ordinary Differential Equations
Final Examination – Solutions
December 17, 2004

I. A) The equation is separable, so separating and integrating,

$$\int \frac{dy}{(y-2)^{1/3}} = \int cos(t)\ dt$$
$$\frac{3}{2}(y-2)^{2/3} = \sin(t) + c$$
$$y = \left(\frac{2}{3}\sin(t) + c\right)^{3/2} + 2$$

From the initial condition $y(0) = 3$, we get $c = 1$.

B) In this equation, $f(t, y) = \cos(t)(y-2)^{2/3}$ is continuous on the whole t, y-plane. But $\frac{\partial f}{\partial y} = \frac{2}{3}\cos(t)(y-2)^{-1/3}$ is not continuous at $(0, 2)$. So the hypotheses of the Existence and Uniqueness Theorem are not satisfied. $y(t) \equiv 2$ is one solution since $y'(t) = 0 = \cos(t)(2-2)^{2/3}$. But there are other solutions as well: For instance taking $c = 0$ in solution from A gives another.

II. Equilibrium points are $x = \pm\sqrt{a}$ for $a \geq 0$ (none for $a < 0$). The phase lines for $a \leq 0$ should have all arrows pointing up, for $a > 0$, the arrows point up if $x > \sqrt{a}$ or if $x < -\sqrt{a}$. They point down between $\sqrt{a}$ and $-\sqrt{a}$. So $x = \sqrt{a}$ is a source, and $x = -\sqrt{a}$ is a sink.

III. A) Taking t in hours,

$$A(t) = 15\cos\left(\frac{\pi t}{12}\right) + 55$$

B) The ODE $y' = -1.5y + 1.5A(t)$ is first order linear. Using entry # 9 in the table of integrals we have

$$y(t) = ce^{-1.5t} + 55 + \frac{22.5}{(\pi/12)^2 + (1.5)^2}\left(1.5\cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right)\right)$$

C) The term containing the c goes to zero as $t \to +\infty$. So that term does not affect the long-term behavior of the solution. (This is like the transient terms in the solutions of our 2nd order forced oscillator equations).

IV. A) For this family of 1st order systems, $T = a$ and $D = -4a$. The family follows the line $D = -4T$ in the trace-determinant plane. That line crosses the parabola $D = T^2/4$ at $T = -16$ and $T = 0$. So, the systems have spiral sinks for $-16 < a < 0$.

B) With $a = 2$, the characteristic polynomial is $\lambda^2 - 2\lambda + 8$, so the eigenvalues are $\lambda = 4, -2$. The origin is a saddle point in the phase portrait. The $\lambda = 4$ line is spanned by the eigenvector $(4, 1)$, and the $\lambda = -2$ line is spanned by the eigenvector $(-2, 1)$.

V. Expanding along row 3, the characteristic polynomial is $(-3-\lambda)(2-\lambda)^2$. The dimension of the kernel of

$$A - 2I = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

is just 1; the kernel is spanned by $(0, 1, 0)$. An eigenvector for $\lambda = -3$ is $(-5, 1, 25)$. So the canonical form is

$$C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

To find the change of basis matrix here we want a vector V_2 such that $(A - 2I)V_2 = V_1 = (0, 1, 0)$. By inspection of the matrix $A - 2I$, we see that $V_2 = e_1$ will work. So

$$Q = \begin{pmatrix} 0 & 1 & -5 \\ 1 & 0 & 1 \\ 0 & 0 & 25 \end{pmatrix}$$

Then the solution of the canonical form system $Y' = CY$ is

$$Y = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the solution of $X' = AX$ is

$$X = QY = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} -5 \\ 1 \\ 25 \end{pmatrix}$$

VI. A) Characteristic polynomial is $\lambda^2 + 6\lambda + 9 = 0$, so $\lambda = -3$ is a repeated root.

$$y = c_1 e^{-3t} + c_2 t e^{-3t}$$

B) Using the particular solution $y_p = Ae^{-t} + Bt^2 + Ct + D$ (since no forcing term is a solution of the homogeneous equation), we get $A = 3/4$, $B = 1/9$, $C = -4/27$, $D = 2/27$.

$$y = c_1 e^{-3t} + c_2 t e^{-3t} + 3e^{-t}/4 + t^2/9 - 4t/27 + 2/27$$

VII. A) From the second equation, at equilibrium either $x = 0$ or $x = 2y - 1$. In the first case, $y = 1$. In the second, $y = 0$ and $x = -1$. So there are two equilibrium points: $(0, 1)$ and $(-1, 0)$.

B) The Jacobian of the system is

$$J(x, y) = \begin{pmatrix} 1 & -1 \\ 2x - 2y + 1 & -2x \end{pmatrix}$$

At $(x, y) = (0, 1)$, $J(0, 1) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ which has determinant -1. So $(0, 1)$ is a *saddle point*. At $(x, y) = (-1, 0)$, $J(-1, 0) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ which has trace 3 and determinant 1. So $(0, 1)$ is a *source* (since $D < T^2/4$ and $T > 0$). In both cases, the equilbrium point of the non-linear system has the same type by our general theorem.

VIII. A) Yes, this system is Hamiltonian, with Hamiltonian function $H = e^{xy}$. Note $H_y = xe^{xy} = f(x, y)$ and $-H_x = -ye^{xy} = g(x, y)$.

B) If a system is Hamiltonian, then the linearization at each critical point has coefficient matrix

$$A = \begin{pmatrix} H_{xy} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}$$

(using equality of mixed partials). The sum of the diagonal entries is zero, so $\mathrm{Tr}(A) = 0$. (Note: it is this fact that tells us that equilibrium points of Hamiltonian systems are always either centers or saddles.)

C) System 1 has a source so it cannot be Hamiltonian. System 3 has a spiral sink, so it cannot be Hamiltonian either. Only system 2 is possibly Hamiltonian.

Extra Credit: See Section 5.4 in Blanchard, Devaney, and Hall. The idea is that if we have a gradient system, then by the same chain rule calculation we did in class for Hamiltonian systems, along a solution,

$$\frac{d}{dt} G(x(t), y(t)) = G_x^2 + G_y^2 \geq 0$$

so G increases along solutions. This means that there cannot be any closed curves that are solutions. The linearization of a gradient system at an equilibrium has coefficient matrix:

$$A = \begin{pmatrix} G_{xx} & G_{xy} \\ G_{xy} & G_{yy} \end{pmatrix}$$

This is always a symmetric matrix, so the eigenvalues are real (by the Spectral Theorem from Linear Algebra). The equilbria can be saddles, sinks, or sources, but no “spiral” behavior is possible for gradient systems.