Mathematics 304, section 2 – Ordinary Differential Equations
Review Sheet – Exam 1, Selected Answers
September 30, 2004

C) Sketch the phase line for each equation, identify equilibria as sinks, sources, or neither. Then for the given initial conditions, sketch qualitative graphs of the corresponding solutions:

$y' = y\cos y$, $y(0) = \pi, 0, 5\pi/3$

Answers: Equilbrium points at $y = 0$ and all solutions of $\cos(y) = 0$, that is all odd integer multiples of $\pi/2$:

$$y = 0, (2k+1)\pi/2, \text{all} k \in \mathbf{Z}$$

By considering the sign of $\cos(y)$, we can see that the sources and sinks *alternate*:

$$\ldots, -3\pi/2, 0, 3\pi/2, \ldots$$

are sources, and

$$\ldots, -5\pi/2, -\pi/2, \pi/2, 5\pi/2, \ldots$$

are sinks. (These can also be seen by using our derivative criterion: For instance, if $f(y) = y\cos(y)$, then $f'(y) = \cos(y) - y\sin(y)$. So for instance $f'(\pi/2) = \cos(\pi/2) - \pi/2\sin(\pi/2) = -\pi/2 < 0$. Hence $y = \pi/2$ is a sink.)

The solution with $y(0) = \pi$ decreases toward the sink at $\pi/2$, the solution with $y(0) = 0$ is the equilibrium solution $y(t) = 0$ all t. The solution with $y(0) = 5\pi/3$ increases toward the sink at $5\pi/2$ since the initial point lies above the source at $y = 3\pi/2$.

D) Sketch the phase line for the ODE

$$y' = \frac{1}{(y-2)(y+1)}$$

and discuss the behavior of the solution with $y(0) = 1/2$.

Answer: The right side has vertical asymptotes at $y = 2, -1$. Between the two asymptotes $1/((y-2)(y+1)) < 0$, so the solution with $y(0) = 1/2$ is decreasing.

E) Sketch the bifurcation diagram for the family of equations

$$y' = y(1-y)^2 + a$$

and describe the bifurcations that occur as a increases from $-\infty$ to $+\infty$.

With $a = 0$, the graph $z = y(1-y)^2$ crosses the y-axis at $y = 0$ and it is tangent to the y-axis at the double root $y = 1$ (a local min). It is a standard cubic polynomial graph with a local max and local min. The coefficient of y^3 is positive, so it starts negative for $y << 0$ and ends positive for $y >> 0$. The effect of the a if $a \neq 0$ is to shift this graph up or down. If $a > 0$, then the double root at $y = 1$ disappears and there is just one negative root. If $a < 0$, then the graph shifts down. For a range of negative values $a_0 < a < 0$, there are three equilibrium points. We get a negative double root eventually at some $a_0 < 0$. Then if a is very negative ($a < a_0$), the graph shifts down so far that the local max is below the y-axis. From this description you should be able to generate the bifurcation diagram.