Mathematics 304 – Ordinary Differential Equations
Discussion/Lab 3 – The Competitive Life
November 12-15, 2004

Background

In this assignment, we will continue to study a model for the *competition* of two species of organisms in an environment with finite resources. We will also consider the effect of "harvesting" one of the species.

As we discussed last week one general form of competitive species models (without harvesting) looks like this. Call the population levels of the two species x, y (functions of time, t). We will assume that:

- If either species is not present (that is if either $x = 0$ or $y = 0$), then the population of the other changes according to a *logistic growth law*. That is, if $y = 0$ the growth of x is approximately exponential if x is small but there is a finite carrying capacity (maximum sustainable population) for x. So the population of x will tend to level off at some M as $t \to \infty$. Similarly for y if $x = 0$.
- The *interaction* between the two species x and y is described by *damping terms* (negative) in the two equations. If $y > 0$, then there is a damping effect on the growth of x, and vice versa.

The simplest model that satisfies both of these assumptions is a system of 1st order ODE:

$$(1) \qquad \begin{cases} \frac{dx}{dt} = ax(M_1 - x) - bxy \\ \frac{dy}{dt} = cy(M_2 - y) - dxy. \end{cases}$$

The a, b, c, d, M_1, M_2 in (1) are *positive* constants reflecting the properties of the two species individually and their interaction. For example, the $-bxy$ term in the first equation describes the damping effect that a positive value of y has on the growth of x. If $y = 0$, then $\frac{dx}{dt} = ax(M_1 - x)$ is a logistic equation, and M_1 represents the maximum sustainable population (carrying capacity of the environment) for x if $y = 0$. Similarly in the second equation.

Our goal is to determine when it is possible for the two species to *stably coexist*, and conversely what conditions ensure that one species always drives the other to extinction (ultimately, by deriving conditions on the constants a, b, c, d that will say which outcome is predicted by the model). Our tools will be the general results we have developed concerning first order autonomous systems, linearization at a equilibrium point, solutions of first order linear systems with constant coefficients, and so forth.

There will be two portions of the assignment – a preliminary group discussion day today, followed by a lab day next Monday in SW 219 where we will plot some particular examples using Maple and study their behavior.

A) The first question you will need to address is: What should *stable coexistence* mean, in mathematical terms? Certainly coexistence should imply that the populations of the two species stay *constant (or maybe almost constant)* over an extended time interval. The stability should say that the populations will evolve toward coexistence levels if they start at a nearby point in the phase plane. So, in mathematical terms, what are we looking for?

B) For example, consider the following systems of the form (1) representing two different competitive scenarios between two species:

1)
$$\begin{cases} \frac{dx}{dt} = x(2-x) - xy \\ \frac{dy}{dt} = y(3-y) - 2xy \end{cases}$$

2)
$$\begin{cases} \frac{dx}{dt} = 2x(3-x) - xy \\ \frac{dy}{dt} = 2y(3-y) - xy \end{cases}$$

For each system, find the equilibrium points, compute the linearization of the system at each one and determine the type of the equilibrium point by examining eigenvalues and eigenvectors of the linearized system. Then, "put your local pictures at each equilibrium point together" to guess the overall form of the phase portrait (the shapes of the solution curves) for the system.

Here's one additional tool that may help you to visualize what the solutions are doing. The curves in the xy phase plane where $dx/dt = 0$ and $dy/dt = 0$ *separately* are called *nullclines* of the system. At all points on the $dy/dt = 0$ nullcline, $dy/dx = (dy/dt)/(dx/dt) = 0$. What does this say about the solution curve through one of these points? Similarly, at all points on the $dx/dt = 0$ nullcline, $dy/dx = (dy/dt)/(dx/dt)$ is undefined. What does that say about the solution curve through one of those points?

Key Question: Should the 2 competing species be able to *coexist* stably in either of these cases?

C) Find all the equilibrium points of a general system (1) (in terms of the a, b, c, d, M_1, M_2). How many different equilibrium points can systems of the form (1) have? (There are two possibilities – be sure you find them both.) Are *all* of the equilibrium points always relevant for our questions here?

D) From the information you have now, you should be able to answer the following question: Could a pair of species competing according to (1) ever coexist at *several different* of population levels? Or is there just one coexistence population level for each species?

Lab Questions

E) Use the Maple *phaseportrait* command to generate plots of the trajectories for both of the systems from question B above. See the handout from class on November 10 (available on the course homepage) for information on the form of this command. Compare with your theoretical predictions from the linearizations (question B).

F) "Harvesting" one species means removing some number from the population (e.g. as a result of humans hunting that species for food or sport). If the species x is harvested at a constant rate $h > 0$ per unit time, we might change the model (1) to

$$(2) \qquad \begin{cases} \frac{dx}{dt} = ax(M_1 - x) - bxy - h \\ \frac{dy}{dt} = cy(M_2 - y) - dxy) \end{cases}$$

For example consider the case:

$$\begin{cases} \frac{dx}{dt} = x(1 - 4x - y) - h \\ \frac{dy}{dt} = y(1 - 5x - 2y) \end{cases}$$

For $h = 0, 1/32, 5/32$ determine the equilibrium points and discuss their types and stability with our linearization methods. Then use Maple to plot the phase portraits and check your work. In each case, discuss the implications of the model (2) – what happens to each species?

Putting it All Together

G) Write a short essay (two or three paragraphs at most) addressing the following question: When is it possible for two species evolving according to (1) (that is – *without "harvesting"*) to *stably coexist*, and conversely what conditions ensure that one species always drives the other to extinction? What conditions on the constants a, b, c, d, M_1, M_2 say which outcome is predicted by the model? Explain your conclusions by refering to the general calculations from question C and the the examples from questions B and E.

Assignment

Group discussion and lab write-ups due no later than Monday, November 22.