Mathematics 304 – Ordinary Differential Equations
Discussion/Lab 3 – Selected Solutions
November 29, 2004

F. It will save some work here to find the equilibrium points and the Jacobian matrix of the systems:

$$\begin{cases} \frac{dx}{dt} = x - 4x^2 - xy - h \\ \frac{dy}{dt} = y - 2y^2 - 5xy \end{cases}$$

in general (i.e. as functions of h), before looking at the specific h-values given in the problem. First, setting the right-hand side of the second equation equal to zero and factoring, we get

$$y(1 - 2y - 5x) = 0$$

so at the equlibrium points, either $y = 0$, or $y = \frac{-5}{2}x + \frac{1}{2}$.

Substituting $y = 0$ into the first equation we get $4x^2 - x + h = 0$, so solving by the quadratic formula

$$x = \frac{1 \pm \sqrt{1 - 16h}}{8},$$

and we have two equilibrium points

$$(x, y) = \left(\frac{1 \pm \sqrt{1 - 16h}}{8}, 0\right)$$

(as long as $h < 1/16$).

Similarly, substituting $y = \frac{-5}{2}x + \frac{1}{2}$ into the first equation, we get another quadratic equation in x:

$$\frac{3}{2}x^2 - \frac{1}{2}x + h = 0$$

so by the quadratic formula again,

$$x = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 6h}}{3} = \frac{1 \pm \sqrt{1 - 24h}}{6}.$$

We have possibly two equilibrium points from this as well (as long as $h < 1/24$).

The Jacobian matrix of the system is independent of h:

$$J(x, y) = \begin{pmatrix} 1 - 8x - y & -x \\ -5y & 1 - 4y - 5x \end{pmatrix}.$$

Substituting $h = 0$, we have four equilibrium points:

$$(0, 0),\ (1/4, 0),\ (0, 1/2),\ (1/3, -1/3).$$

The last is not in the first quadrant of the (x, y)-plane, however. Hence it does not represent a realistic pair of population values, and we will disregard it. Checking the other three critical points, we find:

1) $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $(0,0)$ is a *source*. (This is always true for these systems with no harvesting term ($h = 0$).)

2) $J(1/4,0) = \begin{pmatrix} -1 & -1/4 \\ 0 & -1/4 \end{pmatrix}$, so $(1/4,0)$ is a *sink* ($tr(J) < 0$ and $det(J) > 0$, so eigenvalues are both negative).

3) $J(0,1/2) = \begin{pmatrix} 1/2 & 0 \\ -5/2 & -1 \end{pmatrix}$, so $(0,1/2)$ is a *saddle point* (eigenvalues have opposite signs).

The phase portrait plot for this system should show all solutions starting from points in the first quadrant tending to $(1/4,0)$ as $t \to +\infty$, unless the initial condition has $x = 0$. Note: This is *not a case of stable coexistence* because $y \to 0$ as $t \to +\infty$. The competition is driving that species to extinction.

Substituting $h = 1/32$, we have four equilibrium points:

$$(\frac{1+\sqrt{1/2}}{8},0) \doteq (.213,0),\ (\frac{1-\sqrt{1/2}}{8},0) \doteq (.037,0)\ .$$
$$(1/4,-1/8),\ (1/12,-7/24)$$

The third one now is not in the first quadrant of the (x,y)-plane, and we will disregard it. Checking the other three critical points, we find:

1) $J(.213,0) \doteq \begin{pmatrix} -.707 & -.213 \\ 0 & -.067 \end{pmatrix}$, so $(.213,0)$ is a *sink* (both eigenvalues negative).

2) $J(.037,0) \doteq \begin{pmatrix} .707 & -.037 \\ 0 & .817 \end{pmatrix}$, so $(.037,0)$ is a *source* (eigenvalues are both positive).

3) $J(1/12,7/24) = \begin{pmatrix} 1/24 & -1/12 \\ -35/24 & -7/12 \end{pmatrix}$, so $(1/12,7/24)$ is a *saddle point* (eigenvalues have opposite signs since determinant is negative).

The phase portrait plot for this system should show solutions starting from points far enough to the right in the first quadrant tending to $(.213,0)$ as $t \to +\infty$ and other solutions starting from points with $x \le .037$ tending toward the y-axis. Note: This is *not a case of stable coexistence* either because either $x \to 0$ or $y \to 0$ as $t \to +\infty$. The competition is driving one or other of the two species to extinction. The one exception would be if the initial condition put us exactly on one of the separatrix curves leading into the saddle at $(1/12,7/24)$, but that is not a stable situation because nearby initial conditions would lead to very different outcomes.

With $h = 5/32$, all of the equilibria found above will have complex coordinates, so we disregard all of them. All solutions for this system tend toward the y-axis as $t \to \infty$. Thus, x is going to zero. The effect of the larger harvesting term is to "shift the terms of the competition" in favor of y.

G. Note: I was actually looking for a lot more here than any of the groups presented in their solutions, so this question was not counted for the total on the assignment.

The main question here was supposed to be: *Under what conditions on a, b, c, d, M_1, M_2 can stable coexistence take place?* From question A, this means: when will the system have an equilibrium point (x_0, y_0) in the interior of the first quadrant (so $x_0, y_0 > 0$) that is a *sink*?

Claim. *This is true provided that* $ac - bd > 0, cM_2 - dM_1 > 0, aM_1 - bM_2 > 0$. *Equivalently, this is true if the fourth equilibrium point of the system lies in the first quadrant, but outside the triangle with vertices at the other three equilibria* $(0,0), (M_1, 0), (0, M_2)$.

Before showing this, we recall the two systems from question B:

- In system 1, the fourth equilbrium was at $(1, 1)$, which is *inside* the triangle with vertices $(0, 0), (2, 0), (0, 3)$. This system had a saddle point at $(1, 1)$, and there was no stable coexistence.
- In system 2, the fourth equilibrium was at $(2, 2)$, which is *outside* the triangle with vertices at $(0, 0), (3, 0), (0, 3)$. In this case, $(2, 2)$ was a sink, which made for a stable coexistence situation at those population levels

Now, we show why the claim is true. From the factorizations of the right-hand sides in the basic system, we know that the fourth equilibrium is the intersection of the lines

$$\begin{cases} ax + by = aM_1 \\ dx + cy = cM_2 \end{cases}$$

or

$$(x_0, y_0) = \left(\frac{c(aM_1 - bM_2)}{ac - bd}, \frac{a(cM_2 - dM_1)}{ac - bd} \right).$$

By the hypotheses in the claim (and the general hypotheses that $a, c > 0$), we see that $x_0 > 0$ and $y_0 > 0$. So the fourth equilibrium is in the interior of the first quadrant. Moreover, computing the Jacobian matrix of the system at (x_0, y_0), we find

$$J(x_0, y_0) = \begin{pmatrix} \frac{-ac(aM_1 - bM_2)}{ac - bd} & \frac{-bc(aM_1 - bM_2)}{ac - bd} \\ \frac{-ad(cM_2 - dM_1)}{ac - bd} & \frac{-ac(cM_2 - dM_1)}{ac - bd} \end{pmatrix}$$

This gives, under the hypotheses of the Claim,

$$\det(J(x_0, y_0)) = \frac{ac}{(ac - bd)}(aM_1 - bM_2)(cM_2 - dM_1) > 0$$

and

$$\operatorname{tr}(J(x_0, y_0)) = \frac{-ac}{(ac - bd)}((aM_1 - bM_2) + (cM_2 - dM_1)) < 0$$

These inequalities show that (x_0, y_0) is a sink.

(It is not difficult to see that the condition $cM_2 - dM_1 > 0$ also implies that $(M_1, 0)$ is a saddle point, and similiarly, $aM_1 - bM_2 > 0$ implies that $(0, M_2)$ is a saddle point.)

To see the equivalence with the second form in the Claim, note that the conditions can be rewritten as

$$\frac{a}{b} > \frac{M_2}{M_1} > \frac{d}{c} > 0$$

or

$$(*) \qquad \frac{-a}{b} < \frac{-M_2}{M_1} < \frac{-d}{c} < 0.$$

Geometrically, $\frac{-a}{b}$ is the slope of the line $ax + by = aM_1$ from the first ODE in the system. Also, $\frac{-M_2}{M_1}$ is the slope of the line between the equilibrium points $(M_1, 0)$ and $(0, M_2)$. Finally, $\frac{-d}{c}$ is the slope of the line $dx + cy = cM_2$ from the second ODE. The inequalities (*) are equivalent to saying that the fourth equilbrium point (x_0, y_0) (the intersection of the two lines from the ODE's) lies outside the triangle with vertices $(0, 0), (M_1, 0), (0, M_2)$.