Mathematics 372 – Numerical Linear Algebra
Selected Solutions for Problem Set 3
February 9, 2007

I C. Let A be a tridiagonal positive-definite symmetric matrix. We want to show that the Cholesky factor R of A is also tridiagonal (and upper-triangular). This means that we want to show that in R,

$$r_{ij} = 0 \quad \text{whenever} \quad i + 2 \leq j \leq n.$$

We can prove this by induction on i, using the "row-by-row" form of the Cholesky factorization discussed in class (see equation (1.4.14) in text). Note that that process automatically produces an upper-triangular R, so the entries below the main diagonal will always be zero.

The base case is $i = 1$. In row 1, the diagonal entry is computed by $r_{11} = \sqrt{a_{11}}$. Then the remainder of the first row entries are

$$r_{1j} = \frac{a_{1j}}{r_{11}}.$$

Since A is tridiagonal, $a_{1j} = 0$ when $j \geq 3$. This shows that $r_{1j} = 0$ when $j \geq 3$ too.

Now assume that we have shown that the first k rows of R have the zeros "in the right places":

$$r_{ij} = 0 \quad \text{whenever} \quad 1 \leq i \leq k \quad \text{and} \quad i + 2 \leq j \leq n.$$

In row $k + 1$, after the diagonal entry, the rest of the row is computed by

$$r_{k+1,j} = \frac{a_{k+1,j} - \sum_{l=1}^{k} r_{l,k+1} r_{l,j}}{r_{k+1,k+1}}.$$

We want to see what happens when $j \geq (k + 1) + 2 = k + 3$. Since A is tridiagonal, $a_{k+1,j} = 0$. Furthermore, if $j \geq k + 3$, then since $l \leq k$, the difference between the column and row numbers in the second factor in the terms in the sum is

$$j - l \geq k + 3 - l \geq k + 3 - k \geq 3.$$

By the induction hypothesis, *all* of the $r_{l,j} = 0$ when $l \leq k$ and $j \geq k + 3$. This means that all the terms on the top of the formula for $r_{k+1,j}$ are zero, so $r_{k+1,j} = 0$. This concludes the proof.

Comment: It's true of course that most of the first factors $r_{l,k+1}$ are zero too. This only fails when $l = k$, but even then the other factor $r_{lj} = 0$.

III B. First we show the claim from Exercise 1.4.35 in the text. This is a case where induction is *not* the best way to go. We assume A has Cholesky factor R and then we consider the upper-left $j \times j$ submatrix A_j in A (the entries from rows 1 through j and columns 1 through j).

(Incidentally, you might wonder how we know A_j is positive-definite. The reason is that we can consider all nonzero vectors with zeroes in entries $j+1$ through n. For any such vector $x = (x_j\ 0)^t$, we have, since A is positive-definite,

$$0 < x^t A x = (x_j^t\ 0) \begin{pmatrix} A_j & B \\ B^t & C \end{pmatrix} \begin{pmatrix} x_j \\ 0 \end{pmatrix} = x_j^t A_j x_j$$

Since x_j can be any nonzero vector in $\mathbf{R}^j$, this implies A_j is also positive-definite.)

Now, the problem is to show that the similarly-located submatrix R_j of R is the Cholesky factor of A_j. That is, we must show that $R_j^t R_j = A_j$, R_j is upper-triangular, and has positive diagonal entries. The second two statements here are automatic because R is upper-triangular with positive diagonal entries. R_j consists of the entries in rows 1 to j and columns 1 to j of R, so R_j is also upper-triangular, and the diagonal entries in R_j are the same as the first j diagonal entries in R. The equation $R_j^t R_j = A_j$ comes by considering the partitioned form of R and A. Write

$$R = \begin{pmatrix} R_j & D \\ 0 & E \end{pmatrix}$$

Then

$$R^t R = \begin{pmatrix} R_j^t & 0 \\ D^t & E^t \end{pmatrix} \begin{pmatrix} R_j & D \\ 0 & E \end{pmatrix} = \begin{pmatrix} A_j & B \\ B^t & C \end{pmatrix}.$$

Thinking about how matrix multiplication works, we see that the entries in the first j rows and columns are $R_j^t R_j = A_j$, since the rest of the entries on the first j rows of R^t are zero. This proves 1.4.35.

Now, to prove the envelope theorem, we want to think of organizing the computation of the Cholesky factor R *column-by-column* rather than row-by-row as in the discussion from class and the proof of I C above. The key point of what we want to show is that if A has zero entries in column j and rows $i = 1$ to row $k(j) - 1$ for some $k(j) < i$, but $a_{k(j),j} \neq 0$, then the same is true for the Cholesky factor R. We argue by induction on the size n of the matrix.

In the base case $n = 1$, there is nothing to prove: $R = (r_{11})$ has the same envelope as $A = (a_{11})$ by definition. So assume the result is true for all $j \times j$ positive-definite symmetric A, and consider A of size $(j+1) \times (j+1)$. We partition the matrix A as in 1.4.35, but making A_j consist of all but the last row and column:

$$A = \begin{pmatrix} A_j & c \\ c^t & a_{j+1,j+1} \end{pmatrix}$$

If R is the Cholesky factor of A, then 1.4.35 says the submatrix R_j of R is the Cholesky factor of A_j, so by induction those matrices have the same envelopes.

We only need to consider what happens in the final column. So write

$$R = \begin{pmatrix} R_j & h \\ 0 & r_{j+1,j+1} \end{pmatrix}.$$

Then

$$R^t R = \begin{pmatrix} R_j^t & 0 \\ h^t & r_{j+1,j+1} \end{pmatrix} \begin{pmatrix} R_j & h \\ 0 & r_{j+1,j+1} \end{pmatrix} = \begin{pmatrix} R_j^t R_j & R_j^t h \\ h^t R_j & h^t h + r_{j+1,j+1}^2 \end{pmatrix} = \begin{pmatrix} A_j & c \\ c^t & a_{j+1,j+1} \end{pmatrix}$$

This shows

$$R_j^t h = c$$
$$h^t h + r_{j+1,j+1}^2 = a_{j+1,j+1}$$

Suppose the first nonzero entry in c (down from the top of the matrix) is in row k. We solve for h from the equation $R_j^t h = c$ by forward substitution:

$$\begin{aligned} r_{11} h_1 &= 0 \\ r_{12} h_1 + r_{22} h_2 &= 0 \\ &\vdots \\ r_{1,k-1} h_1 + \cdots + r_{k-1,k-1} h_{k-1} &= 0 \\ r_{1k} h_1 + \cdots + r_{kk} h_k &= c_k \neq 0 \end{aligned}$$

Then $h_1 = h_2 = \cdots = h_{k-1} = 0$, but then $h_{kk} = c_k / r_{kk} \neq 0$. This shows that R has its first nonzero in the same location in column $j+1$ as A does, so the envelopes are the same because the parts of the envelopes in columns 1 through j also work by the induction hypothesis. (*Comment*: This is a very neat proof!!)