I C. Let $A$ be a tridiagonal positive-definite symmetric matrix. We want to show that the
Cholesky factor $R$ of $A$ is also tridiagonal (and upper-triangular). This means that we
want to show that in $R$,

$$r_{ij} = 0 \text{ whenever } i + 2 \leq j \leq n.$$ 

We can prove this by induction on $i$, using the “row-by-row” form of the Cholesky factoriza-
tion discussed in class (see equation (1.4.14) in text). Note that that process automatically
produces an upper-triangular $R$, so the entries below the main diagonal will always be zero.
The base case is $i = 1$. In row 1, the diagonal entry is computed by $r_{11} = \sqrt{a_{11}}$. Then
the remainder of the first row entries are

$$r_{1j} = \frac{a_{1j}}{r_{11}}.$$ 

Since $A$ is tridiagonal, $a_{1j} = 0$ when $j \geq 3$. This shows that $r_{1j} = 0$ when $j \geq 3$ too.

Now assume that we have shown that the first $k$ rows of $R$ have the zeros “in the right
places”:

$$r_{ij} = 0 \text{ whenever } 1 \leq i \leq k \text{ and } i + 2 \leq j \leq n.$$ 

In row $k + 1$, after the diagonal entry, the rest of the row is computed by

$$r_{k+1,j} = \frac{a_{k+1,j} - \sum_{l=1}^{k} r_{l,k+1} r_{l,j}}{r_{k+1,k+1}}.$$ 

We want to see what happens when $j \geq (k + 1) + 2 = k + 3$. Since $A$ is tridiagonal,
$a_{k+1,j} = 0$. Furthermore, if $j \geq k + 3$, then since $l \leq k$, the difference between the column
and row numbers in the second factor in the terms in the sum is

$$j - l \geq k + 3 - l \geq k + 3 - k \geq 3.$$ 

By the induction hypothesis, all of the $r_{lj} = 0$ when $l \leq k$ and $j \geq k + 3$. This means that
all the terms on the top of the formula for $r_{k+1,j}$ are zero, so $r_{k+1,j} = 0$. This concludes
the proof.

Comment: It’s true of course that most of the first factors $r_{l,k+1}$ are zero too. This only
fails when $l = k$, but even then the other factor $r_{lj} = 0$.

III B. First we show the claim from Exercise 1.4.35 in the text. This is a case where
induction is not the best way to go. We assume $A$ has Cholesky factor $R$ and then we
consider the upper-left $j \times j$ submatrix $A_j$ in $A$ (the entries from rows 1 through $j$ and
columns 1 through $j$).
(Incidentally, you might wonder how we know $A_j$ is positive-definite. The reason is that we can consider all nonzero vectors with zeroes in entries $j + 1$ through $n$. For any such vector $x = (x_j \ 0)^t$, we have, since $A$ is positive-definite,

$$0 < x^tAx = (x_j \ 0)^t \begin{pmatrix} A_j & B \\ B^t & C \end{pmatrix} \begin{pmatrix} x_j \\ 0 \end{pmatrix} = x_j^tA_jx_j$$

Since $x_j$ can be any nonzero vector in $\mathbb{R}^j$, this implies $A_j$ is also positive-definite. 

Now, the problem is to show that the similarly-located submatrix $R_j$ of $R$ is the Cholesky factor of $A_j$. That is, we must show that $R_j^tR_j = A_j$, $R_j$ is upper-triangular, and has positive diagonal entries. The second two statements here are automatic because $R$ is upper-triangular with positive diagonal entries. $R_j$ consists of the entries in rows 1 to $j$ and columns 1 to $j$ of $R$, so $R_j$ is also upper-triangular, and the diagonal entries in $R_j$ are the same as the first $j$ diagonal entries in $R$. The equation $R_j^tR_j = A_j$ comes by considering the partitioned form of $R$ and $A$. Write

$$R = \begin{pmatrix} R_j & D \\ 0 & E \end{pmatrix}$$

Then

$$R_j^tR_j = \begin{pmatrix} R_j^t & 0 \\ D^t & E^t \end{pmatrix} \begin{pmatrix} R_j & D \\ 0 & E \end{pmatrix} = \begin{pmatrix} A_j & B \\ B^t & C \end{pmatrix}.$$
Then

\[
R^t R = \begin{pmatrix}
R^t_{11} & 0 \\
R^t_{j+1,j+1} & R_{j+1,j+1}
\end{pmatrix}
\begin{pmatrix}
R_{11} & h \\
0 & R_{j+1,j+1}
\end{pmatrix}
= \begin{pmatrix}
R^t_{11}R_{j+1,j+1} & R^t_{j+1,j+1} \\
R^t_{j+1,j+1}R_{j+1,j+1} & R_{j+1,j+1}
\end{pmatrix}
= \begin{pmatrix}
A^t_{j+1,j+1} & c \\
c & a_{j+1,j+1}
\end{pmatrix}
\]

This shows

\[
R^t_{j+1,j+1} = c
\]

\[
h^t h + r_{j+1,j+1}^2 = a_{j+1,j+1}
\]

Suppose the first nonzero entry in \( c \) (down from the top of the matrix) is in row \( k \). We solve for \( h \) from the equation \( R^t_{j+1,j+1} h = c \) by forward substitution:

\[
\begin{align*}
r_{11}h_1 &= 0 \\
r_{12}h_1 + r_{22}h_2 &= 0 \\
&\vdots \\
r_{1,k-1}h_1 + \cdots + r_{k-1,k-1}h_{k-1} &= 0 \\
r_{1k}h_1 + \cdots + r_{kk}h_k &= c_k \neq 0
\end{align*}
\]

Then \( h_1 = h_2 = \cdots = h_{k-1} = 0 \), but then \( h_{kk} = c_k/r_{kk} \neq 0 \). This shows that \( R \) has its first nonzero in the same location in column \( j+1 \) as \( A \) does, so the envelopes are the same because the parts of the envelopes in columns 1 through \( j \) also work by the induction hypothesis. (\textit{Comment: This is a very neat proof!!})