Mathematics 372 – Numerical Linear Algebra
Problem Set 2 – *due:* February 2, 2007

I. For this problem, you will work in a *decimal* floating point number system, where exactly 5 decimal digits are stored, and exponents lie in the range $-12 \leq e \leq 12$. All nonzero numbers will be represented in normalized form as

$$\pm .d_1 d_2 d_3 d_4 d_5 \times 10^e,$$

where $d_i \in \{0, 1, 2, \ldots, 9\}$ and $d_1 \neq 0$. All computations are done with *rounding* to 5 digits.

A) Compute $(.32113 \times 10^3) - (.97532 \times 10^{-1})$
B) Compute $(.63784 \times 10^4)/(.56734 \times 10^8)$
C) What are the largest and smallest numbers that can be represented in normalized form?
D) For the purposes of this part, let's say the system also contains the one non-normalized number 0 as well as all the numbers in normalized form. How many *different* numbers in all are there in this system?
E) What can you say about the *spacing* between "consecutive" numbers in the system? Is it always the same?

II.
A) Show that if $A = (a_{ij})$ is a lower-triangular $n \times n$ matrix, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$. (Hint: Use induction on n.)
B) Show the same formula for $\det(A)$ if A is upper-triangular. (Hint: recall that a determinant can be expanded along any row or column of the matrix, if you have the correct signs.)

III. Do problem 1.3.4 from the text, but changing the right hand side to

$$(3 \quad 4 \quad 6 \quad 5)^t.$$

Do the arithmetic in two different ways:
A) "By hand" using exact rational number ("fraction") arithmetic.
B) "By hand (with calculator)", but using the floating point number system from problem I above. To make effects of this type of arithmetic more apparent, when you do a computation using several additions, multiplications, etc., group the terms and do the rounding to 5 digits and normalization after *each* addition, multiplication, etc. For instance if you are computing $((a * b) + c)/d$, you would first multiply a and b, round that result and normalize, then add c and round and normalize that result, then divide by d and round and normalize again.
C) Compare your results in parts A and B.

IV. Do problem 1.3.17a from the text, but change the right hand side to

$$(1 \quad -3 \quad 5 \quad 5)^t.$$

Do the calculations in the same two ways as in the last problem, and compare your results again.

V. Let A be a real symmetric $n \times n$ matrix. In this problem you will show that A is positive definite if and only if all the eigenvalues of A are strictly positive. You will need to use the following version of the Spectral Theorem:

There is an invertible matrix Q with $Q^t = Q^{-1}$ and

$$A = Q^t \mathrm{diag}(\lambda_1, \ldots, \lambda_n) Q$$

where "diag" means the diagonal matrix with those entries on the main diagonal. (Indeed, Q^t here is the matrix whose columns are the orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of A, but you should not need to use that fact.)

A) Show that if $\lambda_i > 0$ for all i, then A is positive definite.
B) Show conversely that if A is positive definite, then $\lambda_i > 0$ for all i.