Mathematics 372 – Numerical Linear Algebra
Midterm Problem Set
March 16, 2007, *due*: March 23, 2007

General Information and Ground Rules

This is an open-book, open notes, open Swords 335 assignment. You must work *independently* on these problems, and you may not consult with other faculty members or students. You may consult other books, sources in the library, or information from the Internet, but if you find something relevant, you should *footnote* it for proper acknowledgment of sources. When you have finished your solutions, please write out a short statement that you have abided by these ground rules, sign it, and hand it in with your work.

I. (More on Gaussian elimination and LU factorizations).

A) (10) Show that an $n \times n$ matrix A has an LU factorization (equivalently, Gaussian elimination is possible with no row interchanges) if and only if for each $1 \leq k \leq n$, the upper left corner $k \times k$ submatrix

$$A_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$$

is nonsingular.

B) (10) Suppose that A is a matrix that satisfies the condition in part A, and for any k, $1 \leq k \leq n$, partition A as

$$A = \begin{pmatrix} A_k & B \\ C & D \end{pmatrix},$$

where B is $k \times (n-k)$, C is $(n-k) \times k$, and D is $(n-k) \times (n-k)$. Show that there is a unique $(n-k) \times k$ matrix M such that

$$\begin{pmatrix} I_k & 0 \\ -M & I_{n-k} \end{pmatrix} \begin{pmatrix} A_k & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_k & \hat{B} \\ 0 & \hat{D} \end{pmatrix}$$

for some matrices $\hat{B}$ and $\hat{D}$. Also show that $\hat{D} = D - CA_k^{-1}B$.

C) (5) Deduce that A has the *block LU decomposition*

$$A = \begin{pmatrix} I_k & 0 \\ M & I_{n-k} \end{pmatrix} \begin{pmatrix} A_k & \hat{B} \\ 0 & \hat{D} \end{pmatrix}.$$

D) (5) By definition, the matrix A_k satisfies the condition in part A. It can be shown that $\hat{D}$ does as well. Hence each of these matrices also has an LU-decomposition:

$A_k = L_1 U_1$, and $\hat{D} = L_2 U_2$. Express the LU-decomposition of A in terms of L_1, L_2, M, U_1, U_2, and B.

II. (The matrix 2-norm, revisited.) By the general theory of the matrix operator norms associated to vector norms on $\mathbf{R}^n$, the matrix 2-norm of $A \in M_{n\times n}(\mathbf{R})$ is given by

$$\|A\|_2 = \max_{\|u\|_2=1} \|Au\|_2. \tag{1}$$

Your goal in this problem is to prove the formula

$$\|A\|_2 = \sqrt{\max\{|\lambda| : \lambda \text{ is an eigenvalue of } A^t A\}} \tag{2}$$

that we mentioned in class. The equation (1) shows that we need to solve a *constrained optimization problem* to determine $\|A\|_2$.

A) (5) What standard method from multivariable calculus applies to solve constrained optimization problems of the following form:

$$\begin{aligned} \text{maximize} &: f(x_1, \ldots, x_n) \\ \text{subject to} &: g(x_1, \ldots, x_n) = 0? \end{aligned}$$

Describe how the method works in general.

B) (15) For our purposes in (1), it will be most convenient to square both the function $\|Au\|_2$ we seek to maximize and the constraint equation $\|u\|_2 = 1$. Writing $u = (u_1, \ldots, u_n)^t$, apply the method with $f(u_1, \ldots, u_n) = \|Au\|_2^2$ and $g(u_1, \ldots, u_n) = u_1^2 + \cdots + u_n^2 - 1$. Show that the equations for the constrained maximum imply that the vector u where f achieves its maximum must be an eigenvector of $A^t A$, and that the multiplier λ in the method must be the corresponding eigenvalue.

C) (5) Deduce that (2) holds, giving the formula for $\|A\|_2$.

D) (10) Recall the 60×60 adjacency matrix B of the "bucky ball" from Lab/Problem Set 3. Use (2) above and MATLAB to compute $\|B\|_2$. Some technical notes: (a) The MATLAB command `eig` returns a list of all the eigenvalues of a square matrix; (b) the built-in matrix `bucky` is a sparse matrix. If you want to check your answer using the `norm` command, you will need to convert `bucky` to "full" form.

E) (10) If $A = (a_{ij})$, the matrix Frobenius norm is defined by

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2}$$

(note: the absolute values here are unnecessary for real matrices, but necessary if some of the entries in A are complex numbers with nonzero imaginary parts). Show that for all matrices A

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2 \tag{3}$$

F) (5) Compute the Frobenius norm of the bucky ball matrix B using MATLAB and verify that (3) holds in this case.

III. (Roundoff considerations and least squares.) In many situations, the way a problem is set up can have a significant effect on the susceptibility to roundoff errors when we apply standard methods. But fortunately, there are often ways to minimize those effects. For instance, suppose that we want to compute the best-fit line for the following data points

x_i	1.001	1.002	1.003	1.004	1.005	1.006	1.007
y_i	3.664	3.789	3.891	4.022	4.233	5.200	5.329

by solving the normal equations $X^tX\begin{pmatrix} m \\ b \end{pmatrix} = X^tY$ for the least squares problem.

A) (5) Using MATLAB, find the ∞-norm condition number of the coefficient matrix X^tX in the system of normal equations. What does your answer indicate about this system of equations?

B) (10) Our standard way of setting up the normal equations corresponds to using the basis $\{x, 1\}$ for the vector space of linear polynomials. But we could use any other basis we like too. Suppose we use $\{300(x - 1.004), 1\}$ instead (that is, translate and rescale the x_i values). What is the ∞-norm condition number of the coefficient matrix in the normal equations now? Show that the best-fit line can still be computed if we set the problem up this way.

C) (5) Explain your results in A and B geometrically.