8, 10, 8, 8, 8, 12 54 total

MATH 241 - Problem Set 7 Solutions

§4.2 / 3b, 4c, 9

3(b) $f(x,y) = x^2y^2 - x^2y + xy$ $(-1,3)$

1 $f_x = 2xy^2 - 2xy + y$ $f_{xx} = 2y^2 - 2y$ 1

1 $f_y = 2x^2y - x^2 + x$ $f_{xy} = 4xy - 2x + 1$ 1

$f_{yy} = 2x^2$ 1

1st degree: $f(-1,3) + f_x(-1,3)(x+1) + f_y(-1,3)(y-3)$

$= 3 + -9(x+1) + 4(y-3)$ 1

2nd degree $= f(-1,3) + f_x(-1,3)(x+1) + f_y(-1,3)(y-3)$

$+ \frac{1}{2}\left[f_{xx}(-1,3)(x+1)^2 + 2f_{xy}(-1,3)(x+1)(y-3) + f_{yy}(-1,3)(y-3)^2\right]$

$= 3 - 9(x+1) + 4(y-3) + \frac{1}{2}\left[12(x+1)^2 + 2(-9)(x+1)(y-3) + 2(y-3)^2\right]$

$= 3 - 9(x+1) + 4(y-3) + 6(x+1)^2 - 9(x+1)(y-3) + (y-3)^2$

2

4(c) $f(x,y) = \frac{x^4}{4} - x^3 + x^2 + 1 - y^2$

1 $f_x = x^3 - 3x^2 + 2x$ $f_{xx} = 3x^2 - 6x + 2$ 1

1 $f_y = -2y$ $f_{xy} = 0$ 1

$f_{yy} = -2$ 1

critical points:

$\left.\begin{array}{l} x(x-1)(x-2) = 0 \\ y = 0 \end{array}\right\}$ so $(0,0), (1,0), (2,0)$ 2

$(0,0)$: $f_{xx}(0,0) = 2$

$H(0,0) = (2)(-2) - 0^2 = -4$

so this is a saddle point. 1

$(1,0)$: $f_{xx}(1,0) = -1$

$H(1,0) = (-1)(-2) - 0^2 = +2$

so this is a local max 1

$(2,0)$: $f_{xx}(2,0) = 2$

$H(2,0) = (2)(-2) - 0^2 = -4$

saddle point 1

9. (a) $f_x = 3a_1x^2 + 4a_2xy + a_3y^2 + 2b_1x + b_2y + c_1$

$f_y = a_2x^2 + 2a_3xy + 3a_4y^2 + b_2x + 2b_3y + c_2$

$f_{xx} = 6a_1x + 4a_2y + 2b_1$

$f_{xy} = 4a_2x + 2a_3y + b_2$

$f_{yy} = 2a_3x + 6a_4y + 2b_3$

linear (1st degree poly): $d + c_1x + c_2y$

7 quadratic (2nd degree poly): $d + c_1x + c_2y + \frac{1}{2}[2b_1x^2 + 2b_2xy + 2b_3y^2]$

$= d + c_1x + c_2y + b_1x^2 + b_2xy + b_3y^2$

(b) the linear polynomial consists of the terms of total degree ≤ 1 in f. The quadratic polynomial consists of the terms of degree ≤ 2 in f. 1

§ 4.3 /

2 (c) $f(x,y) = xy$ and $g(x,y) = 3x^2 - 2xy + y^2 = 4$

Lagrange equations:

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial g}{\partial x} = 6x - 2y$$
$$\frac{\partial f}{\partial y} = x \qquad \frac{\partial g}{\partial y} = -2x + 2y$$

So we want to solve:

$$\left\{\begin{array}{l} y = \lambda(6x - 2y) \\ x = \lambda(-2x + 2y) \\ 3x^2 - 2xy + y^2 = 4 \end{array}\right\}$$

2

Eliminating λ between 1st two equations:

$$y(-2x + 2y) = x(6x - 2y)$$
$$2y^2 = 6x^2$$
$$\text{or } y^2 = 3x^2$$

- If $y = \sqrt{3}x$ the constraint gives $(6 - 2\sqrt{3})x^2 = 4$,
  so $x = \pm \frac{2}{\sqrt{6 - 2\sqrt{3}}} \doteq \pm 1.256$ $\quad y \doteq \pm 2.175$ $\left(= \frac{\pm 2\sqrt{3}}{\sqrt{6 - 2\sqrt{3}}}\right)$
- If $y = -\sqrt{3}x$, the constraint gives $(6 + 2\sqrt{3})x^2 = 4$
  so $x = \pm \frac{2}{\sqrt{6 + 2\sqrt{3}}} \doteq \pm .650$ $\quad y = \mp \frac{2\sqrt{3}}{\sqrt{6 + 2\sqrt{3}}}$

  $= \mp 1.126$

4

$$f\left(\pm \frac{2}{\sqrt{6 - 2\sqrt{3}}}, \pm \frac{2\sqrt{3}}{\sqrt{6 - 2\sqrt{3}}}\right) = \frac{4\sqrt{3}}{6 - 2\sqrt{3}} \quad \text{max}$$

2

$$f\left(\pm \frac{2}{\sqrt{6 + 2\sqrt{3}}}, \mp \frac{2\sqrt{3}}{\sqrt{6 + 2\sqrt{3}}}\right) = \frac{-4\sqrt{3}}{6 + 2\sqrt{3}} \quad \text{min}$$

(d) $f(x,y) = (x-1)^2 + 4y^2$ $\qquad g(x,y) = 2x^2 + y^2 = 3$

Lagrange Equations:

$$2(x-1) = 4\lambda x$$
$$8y = 2\lambda y$$
$$2x^2 + y^2 = 3$$

(2)

the second equation says either $y = 0$ or $\lambda = 4$

If $y=0$ the constraint gives $x = \pm\sqrt{\frac{3}{2}}$

If $\lambda = 4$, the first equation says $2x - 2 = 16x$, so $x = \frac{-1}{7}$

and $y^2 = 3 - \frac{2}{49}$ $\qquad y = \pm\sqrt{\frac{145}{49}} = \pm\frac{\sqrt{145}}{7}$

$$f\left(\sqrt{\tfrac{3}{2}}, 0\right) = \left(\sqrt{\tfrac{3}{2}} - 1\right)^2 \doteq .0505 \leftarrow \text{min}$$

(4)

$$f\left(-\sqrt{\tfrac{3}{2}}, 0\right) = \left(-\sqrt{\tfrac{3}{2}} - 1\right)^2 \doteq 4.949$$

(2)

$$f\left(-\tfrac{1}{7}, \tfrac{\sqrt{145}}{7}\right) = \left(-\tfrac{1}{7} - 1\right)^2 + \frac{4 \cdot 145}{49} \doteq 13.143$$

max

$$f\left(-\tfrac{1}{7}, -\tfrac{\sqrt{145}}{7}\right) = \left(-\tfrac{1}{7} - 1\right)^2 + \frac{4 \cdot 145}{49} \doteq 13.143$$

3 (c) $f(x,y) = x^2 - 3y^2$ on $S = \{(x,y) \mid x^2 + \frac{y^2}{9} \leq 1,\ x \leq 0\}$

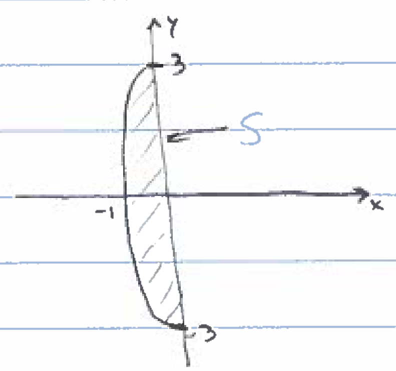
$f_x = 2x$

$f_y = -6y$

(2)

So the only critical point of f

is at $(0,0)$ (on boundary of S)



on $x = 0$: Lagrange equations

$$\begin{cases} 2x = \lambda \cdot 1 \\ -6y = \lambda \cdot 0 \end{cases} \Rightarrow \boxed{y = 0} \qquad (0,0)$$

4

on $x^2 + \frac{y^2}{9} = 1$: Lagrange

$$\begin{cases} 2x = \lambda \cdot 2x & \Rightarrow x = 0 \text{ or } \lambda = 1 \\ -6y = \lambda \cdot \frac{2y}{9} \\ x^2 + y^2/9 = 1 \end{cases}$$

4

$x = 0 \Rightarrow y = \pm 3$

$\lambda = 1 \Rightarrow -6y = \frac{2y}{9}$, so $\boxed{y = 0}$, $\boxed{x = -1}$

$f(0,0) = 0$

$f(0, \pm 3) = -27 \longleftarrow$ min.

2

$f(-1, 0) = 1 \longleftarrow$ max