4, 4, 6, 6; 10, 14, 24   68 total

Math 241 - Problem Set 6 Solutions

§ 3.5

4

3(a) $f(x,y) = xy^2 - y$ $(x_0, y_0) = (1,2)$ $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

1   1

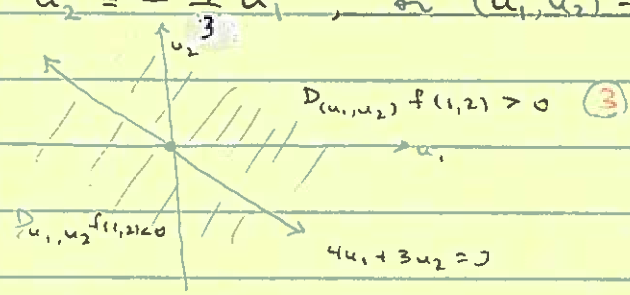
$\frac{\partial f}{\partial x} = y^2$ and $\frac{\partial f}{\partial y} = 2xy - 1$ are continuous on all of $\mathbb{R}^2$, so Proposition 3.3 $\Rightarrow$ f is differentiable at $(1,2)$, and

$$D_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} f(1,2) = \nabla f(1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$= (4, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$= \boxed{\frac{7}{\sqrt{2}}} \quad 2$$

4

4(a) $D_{(u_1,u_2)} f(1,2) = (4,3) \cdot (u_1, u_2)$
$= 4u_1 + 3u_2$ 1

this is $= 0$ for $u_2 = -\frac{4}{3}u_1$, or $(u_1, u_2) = \pm\left(\frac{3}{5}, \frac{-4}{5}\right)$



§ 4.1

6

1 (a) $f(x,y) = x e^{-x^2 - y^2}$

$$\frac{\partial f}{\partial x} = e^{-x^2-y^2}(1 - 2x^2) \quad 1$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2} \cdot (-2xy) \quad 1$$

critical points are $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$ 2

$$f\left(\tfrac{1}{\sqrt{2}}, 0\right) = \tfrac{1}{\sqrt{2}} e^{-1/2} > 0$$

$$f\left(-\tfrac{1}{\sqrt{2}}, 0\right) = -\tfrac{1}{\sqrt{2}} e^{-1/2} < 0$$

2

so $f\left(\frac{1}{\sqrt{2}}, 0\right)$ must be a local and global maximum

$f\left(-\frac{1}{\sqrt{2}}, 0\right)$ " " " " " global minimum

(b) $f(x,y) = x^4 + y^3 - 3y$

$$\frac{\partial f}{\partial x} = 4x^3 \quad 1$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3 \quad 1$$

Critical points are $(0,1)$ and $(0,-1)$ 2

From the contour plot, $f(0,1) = -2$ is a local minimum, and $f(0,-1) = 2$ is neither (a saddle point). The local minimum at $(0,1)$ is not a global minimum since $f(0,-3) = -18 < f(0,1)$

2

2. (d) $f(x,y) = (2x^2 + 3y^2)\, e^{-x^2-y^2}$

10

$$\frac{\partial f}{\partial x} = (4x - 4x^3 - 6xy^2)\, e^{-x^2-y^2} \quad 1$$

$$\frac{\partial f}{\partial y} = (6y - 4x^2y - 6y^3)\, e^{-x^2-y^2} \quad 1$$

critical points:
$$\begin{cases} x(4 - 4x^2 - 6y^2) = 0 \\ y(6 - 4x^2 - 6y^2) = 0 \end{cases} \quad 8$$

From first, $x = 0$ or $4x^2 + 6y^2 = 4$

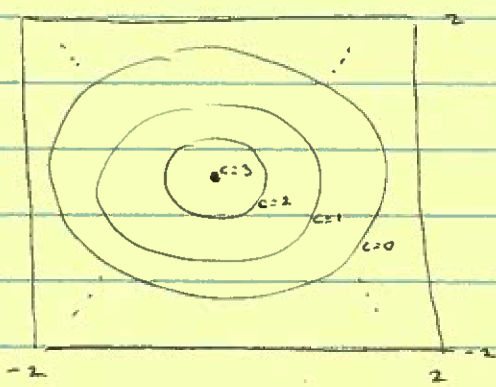
If $x = 0$, second equation gives $y = 0$ or $y = \pm 1$

If $4x^2+6y^2=4$, then second gives $y=0$, so
$x=\pm 1, 0$

Critical points: $(0,0)$, $(0,1)$, $(0,-1)$ 4
$(1,0)$ $(-1,0)$

From the contour plot and the values $f(0,0)=0$,
$f(0,\pm1) = 3e^{-1}$, $f(\pm1,0) = 2e^{-1}$, it
is clear that f has a local and global
minimum at $(0,0)$, local and global maxima 4
at $(0,\pm1)$, and saddle points at $(\pm1,0)$.

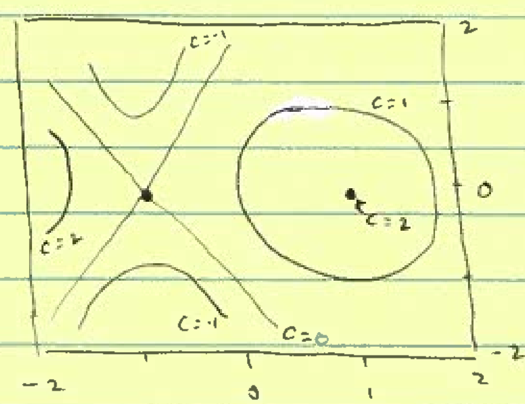
3. (a)



2

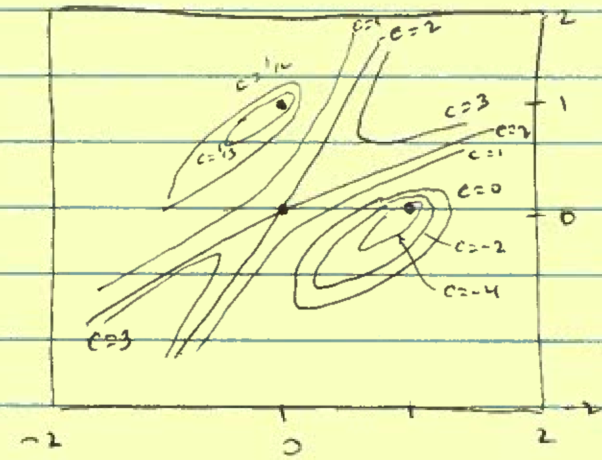
14

(b)



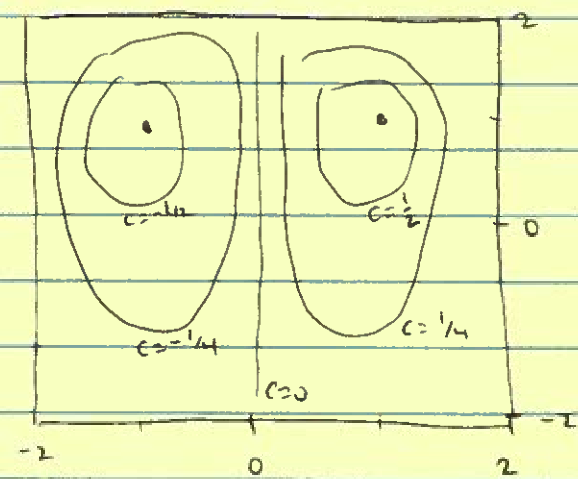
4

(c)



(4)

(d) like Figure 4.5(a)



(4)

anything "close" is OK on these.

4. (a) $f(x,y) = 3x - x^3 - 3xy^2$

24

$$\frac{\partial f}{\partial x} = 3 - 3x^2 - 3y^2 = 0$$

$$\frac{\partial f}{\partial y} = -6xy = 0$$

(2)

second equation: $x = 0$ or $y = 0$

if $x = 0$ first gives $y = \pm 1$

$y = 0$ first gives $x = \pm 1$

$(0,1)$: saddle point of ∇f and f

$(0,-1)$: " " " " " "

4 $(1,0)$: sink of ∇f, local maximum of f

$(-1,0)$: source of ∇f, " minimum of f

(b) $f(x,y) = y^4 - 4y^2 - x^2$

$$\frac{\partial f}{\partial x} = -2x$$

2

$$\frac{\partial f}{\partial y} = 4y^3 - 8y = 4y(y^2-2)$$

Critical points: $(0,0)$, $(0,\sqrt{2})$, $(0,-\sqrt{2})$

$(0,0)$: sink of ∇f, local maximum of f

4 $(0,\sqrt{2})$: saddle of ∇f and f

$(0,-\sqrt{2})$ saddle of ∇f and f

(c) $f(x,y) = (x^2+y^2)e^{-y}$

$$\frac{\partial f}{\partial x} = 2x e^{-y} = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = (2y-(x^2+y^2))e^{-y} = (2y - x^2 - y^2)e^{-y} = 0$$

2

So in second equation $0 = 2y - y^2 = (2-y)y$.

critical points are $(0,0)$, $(0,2)$

4 $(0,0)$: source of ∇f, local minimum of f

$(0,2)$: saddle of ∇f, saddle of f

(d) $\quad f(x,y) = \dfrac{y}{(1+x^2+y^2)}$

$$\frac{\partial f}{\partial x} = \frac{-2xy}{(1+x^2+y^2)^2}$$

2

$$\frac{\partial f}{\partial y} = \frac{1+x^2+y^2-2y^2}{(1+x^2+y^2)^2} = \frac{1+x^2-y^2}{(1+x^2+y^2)^2}$$

$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad x = 0$ or $y = 0$

If $x=0$, $\frac{\partial f}{\partial y} = 0 \Rightarrow y = \pm 1$

If $y = 0$, $\frac{\partial f}{\partial y} \neq 0$.

4

So critical points are $(0,1)$, $(0,-1)$

$(0,-1)$ is a source of ∇f, local min of f

$(0,1)$ is a sink of ∇f, local max of f.