✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓
4, 4, 8, 4, 8, 8, 8, 16, 6, 4, 18 Total: 88

2.1: 1c, 3ac, 4, 5ac, 11
2.2: 1bd, 2bd, 3, 6, 7, 10

MATH 241 PS 3 Solutions

§ 2.1

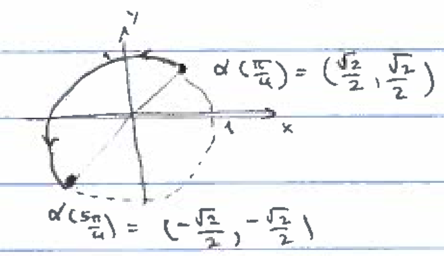
1 (c) the standard parametrization of the line segment from $P = (-2,-2)$ to $Q = (1,6)$ is $\alpha(t) = (-2,-2) + t(3, 8)$ ②

④ with $t \in [0,1]$. the same line segment is obtained for $u \in [3,4]$ if we let $u = t+3$, or $t = u-3$, so

$\beta(u) = (-2,-2) + \underset{t}{(u-3)}(3,8)$ ②, or $\beta(u) = (3u-11, 8u-26)$.

OK if it's $t-3$:)

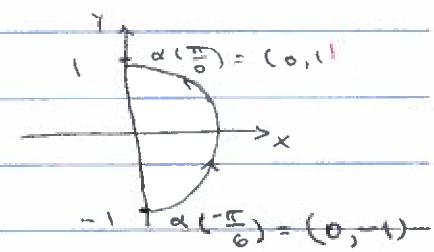
3 (a) $\alpha(t) = (\cos t, \sin t)$ $\quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}$



1 for end points

④ 1 for direction

(c) $\alpha(t) = (\cos 3t, \sin 3t)$ $\quad -\frac{\pi}{6} \le t \le \frac{\pi}{6}$



1 for end points

1 for direction

4. (a) let $\alpha(t) = (x(t), y(t)) = (a \cos t, b \sin t)$ $\quad 0 \le t \le 2\pi$, $a, b > 0$

then $\frac{x(t)^2}{a^2} + \frac{y(t)^2}{b^2} = \left(\frac{x(t)}{a}\right)^2 + \left(\frac{y(t)}{b}\right)^2 = \cos^2 t + \sin^2 t = 1$ all t ④

⑧ this shows the coordinates of α satisfy the given equation

(b) to get $\beta(0) = (0, b)$ we could either use

$\beta(t) = (a \sin(t), b \cos(t))$ (clockwise), or

②

$\beta(t) = (a \cos(t + \frac{\pi}{2}), b \sin(t + \frac{\pi}{2}))$ (counter-clockwise)

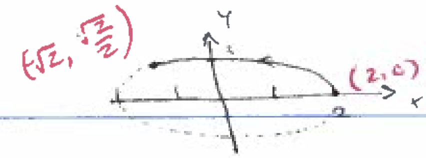
$= (-a \sin(t), b \cos(t))$ (trig identity)

(any of these forms, singly, is OK — no need for all of them.)

(c) $\beta(t) = (a \cos(\pi t), b \sin(\pi t))$ is one

② since $0 \le \pi t \le 2\pi$ for $0 \le t \le 2$.

5. (a) $\alpha(t) = (2\cos t,\ t)$ $\quad 0 \le t \le \frac{3\pi}{4}$



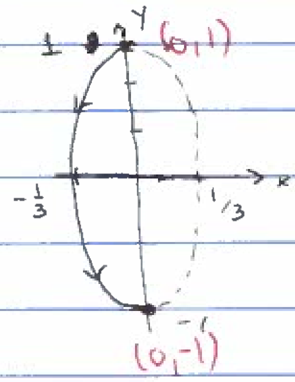
1 for end points

(4) 1 for direction

$\alpha(0) = (2, 0)$

$\alpha(\frac{3\pi}{4}) = (-\sqrt{2}, \frac{\sqrt{2}}{2})$

(c) $\alpha(t) = \left(\frac{1}{3}\cos 2\pi t,\ \sin 2\pi t\right)$, $\quad \frac{1}{4} \le t \le \frac{3}{4}$



$\alpha(\frac{1}{4}) = (\frac{1}{3}\cos\frac{\pi}{2},\ \sin\frac{\pi}{2}) = (0, 1)$

$\alpha(\frac{3}{4}) = (\frac{1}{3}\cos\frac{3\pi}{2},\ \sin(\frac{3\pi}{2})) = (0, -1)$

1 for end points

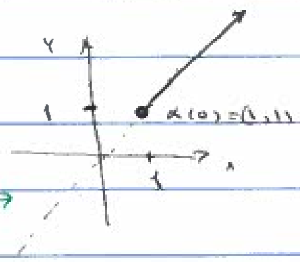
1 for direction

new directions

11. (a) $\alpha(t) = (e^t, e^t)$ lies on the line $y = x$ (or $x = y$) (2)

(8)

this part was not asked for in the problem $\longrightarrow$

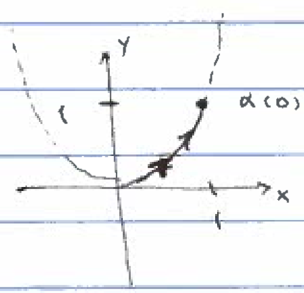


$\lim_{t\to\infty} \alpha(t) = (+\infty, +\infty)$

(b) $\alpha(t) = (e^t, e^{2t})$ lies on the parabola $y = x^2$ (2)

(or $x = \sqrt{y}$, note $x = e^t > 0$

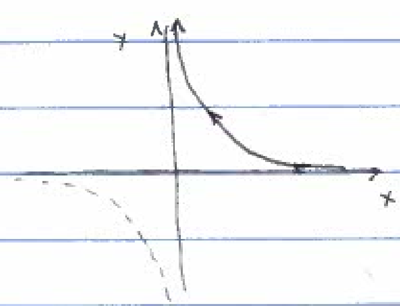
and $y = e^{2t} > 0$ all t)



$\lim_{t\to-\infty} \alpha(t) = (0, 0)$

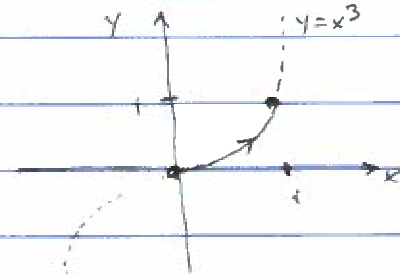
(c) $\alpha(t) = (e^{-t}, e^t)$ lies on $y = \frac{1}{x}$ or $x = \frac{1}{y}$ (2)

the curve is the branch of the hyperbola $xy = 1$ in the 1^{st} quadrant. $\lim_{t\to-\infty}\alpha(t) = (+\infty, 0)$



$\lim_{t\to+\infty} \alpha(t) = (0, +\infty)$

(d) $\alpha(t) = (\sin t, \sin^3 t)$, $0 \le t \le \frac{\pi}{2}$ is the segment of the graph $y = x^3$ (or $x = y^{1/3}$) from $(0,0)$ to $(1,1)$ 2



§ 2.2

1. (b) $\alpha(t) = (2\cos t, 3\sin t)$

8 $\alpha'(t) = (-2\sin t, 3\cos t)$ 3

so $\alpha'(\frac{\pi}{2}) = (-2\sin\frac{\pi}{2}, 3\cos\frac{\pi}{2}) = (-2, 0)$ is the velocity 1

the speed is $\|\alpha'(\frac{\pi}{2})\| = \|(-2,0)\| = 2$ 1

(d) $\alpha(t) = (1, t, \sqrt{1+t^2})$

$\Rightarrow \alpha'(t) = \left(0, 1, \frac{t}{\sqrt{1+t^2}}\right)$ 2

so $\alpha'(0) = (0, 1, 0)$ is the velocity 1

$\|\alpha'(0)\| = \|(0,1,0)\| = 1$ is the speed 1

2. (b) $\alpha(t) = (te^{2t}, e^{t^2}) \Rightarrow \alpha(1) = (e^2, e)$ 1

8 $\alpha'(t) = (e^{2t} + 2te^{2t}, 2te^{t^2}) \Rightarrow \alpha'(1) = (3e^2, 2e)$ 2 1

tangent line: $\beta(u) = \alpha(1) + u\,\alpha'(1)$, $u \in \mathbb{R}$

$(e^2,e) + u(3e^2, 2e)$ $= (e^2 + 3ue^2, e + 2ue)$

$= (e^2(1+3u), e(1+2u))$, $u \in \mathbb{R}$ 2

(d) $\alpha(t) = ((2+\sin 2t)\cos t, (2+2\sin 2t)\sin t, \cos 3t)$

$\therefore \alpha(0) = (2, 0, 1)$ 1

$\alpha'(t) = (-(2+\sin 2t)\sin t + 2\cos 2t\cos t, (2+2\sin 2t)\cos t + 4\cos 2t \sin t, -3\sin$ 2

So $\alpha'(0) = (2,2,0)$. 1

The tangent line is $\beta(u) = \alpha(0) + u\,\alpha'(0)$ 1
$= (2+2u,\ 2u,\ 1)$ $u \in \mathbb{R}$
$(2,0,1) + u(2,2,0)$

3. (a) Let $\alpha(t) = (x(t), y(t), z(t))$, $c \in \mathbb{R}$. Then
16 $c\alpha(t) = (cx(t), cy(t), cz(t))$. Since we assume x', y', z' exist, $(c\alpha(t))' = ((cx(t))', (cy(t))', (cz(t))')$
$= (cx'(t), cy'(t), cz'(t))$
$= c(x'(t), y'(t), z'(t))$
$= c\alpha'(t)$. 4

(b) Let $\alpha(t)$ be as above and $\beta(t) = (u(t), v(t), w(t))$
then $\alpha(t) + \beta(t) = (x(t)+u(t),\ y(t)+v(t),\ z(t)+w(t))$
so $(\alpha(t)+\beta(t))' = ((x(t)+u(t))', (y(t)+v(t))', (z(t)+w(t))')$
$= (x'(t)+u'(t),\ y'(t)+v'(t),\ z'(t)+w'(t))$ (sum rule for derivatives)
$= (x'(t), y'(t), z'(t)) + (u'(t), v'(t), w'(t))$
$= \alpha'(t) + \beta'(t)$ 2

(c) $\alpha(t)\cdot\beta(t)$ is the scalar function
$x(t)u(t) + y(t)v(t) + z(t)w(t)$. By the sum and product rules for derivatives, $(x(t)u(t) + y(t)v(t) + z(t)w(t))'$
$x(t)u'(t) + x'(t)u(t) + y(t)v'(t) + y'(t)v(t) + z(t)w'(t) + z'(t)w(t)$
$= (x(t), y(t), z(t))\cdot(u'(t), v'(t), w'(t)) + (x'(t), y'(t), z'(t))\cdot(u(t), v(t), w(t))$
$= \alpha(t)\cdot\beta'(t) + \alpha'(t)\cdot\beta(t)$ 4

(d) $\alpha(t)\times\beta(t)$ is the vector function
$\alpha(t)\times\beta(t) = (y(t)w(t) - z(t)v(t),\ -(x(t)w(t) - z(t)u(t)),\ x(t)v(t) - y(t)u(t))$
By the sum and product rules for derivatives, 4
$(\alpha(t)\times\beta(t))' = (y(t)w'(t) + y'(t)w(t) - z(t)v'(t) - z'(t)v(t),\ -x(t)w'(t) - x'(t)w(t)$
$+ z(t)u'(t) + z'(t)u(t),\ x(t)v'(t) + x'(t)v(t) - y(t)u'(t) - y'(t)u(t)$

$= \big(y(t)w'(t) - z(t)v'(t),\ -(x(t)w'(t) - z(t)u'(t)),\ x(t)v'(t) - y(t)u'(t) \big)$

$+ \big(y'(t)w(t) - z'(t)v(t),\ -(x'(t)w(t) - z'(t)u(t)),\ x'(t)v(t) - y'(t)u(t) \big)$

$= \alpha(t) \times \beta'(t) + \alpha'(t) \times \beta(t).$

6. (a) $\alpha(t) = (x_0 + ct(x_1 - x_0),\ y_0 + ct(y_1 - y_0),\ z_0 + ct(z_1 - z_0))$

6

$\therefore\ \alpha'(t) = (c(x_1 - x_0),\ c(y_1 - y_0),\ c(z_1 - z_0))$

and $\alpha''(t) = (0, 0, 0)$ 2

(b) $\alpha(t) = (R\cos(ct), R\sin(ct))$

$\therefore\ \alpha'(t) = (-Rc\sin(ct), Rc\cos(ct))$ (chain rule)

and $\alpha''(t) = (-Rc^2\cos(ct), -Rc^2\sin(ct))$ (")

$= -c^2\alpha(t)$ 2

(the acceleration is directed toward $(0,0)$ from $\alpha(t)$)

(c) $\alpha(t) = (R\cos t, R\sin t, ct)$

$\therefore\ \alpha'(t) = (-R\sin t, R\cos t, c)$

and $\alpha''(t) = (-R\cos t, -R\sin t, 0)$ 2

[#7 on separate page following – sorry!]

18 10. (a) To approximate the velocity at $t = 0$ we could use $\alpha'(0) \doteq \dfrac{\alpha(4) - \alpha(0)}{4 - 0} = \dfrac{\alpha(4) - \alpha(0)}{4}$ then 4

Similarly $\alpha'(4k) \doteq \dfrac{\alpha(4(k+1)) - \alpha(4k)}{4(k+1) - 4k} = \dfrac{\alpha(4(k+1)) - \alpha(4k)}{4}$

also ok (better) would be $\frac{1}{2}\left(\frac{\alpha(4) - \alpha(0)}{4} + \frac{\alpha(0) - \alpha(72)}{-4}\right)$ etc.

$\frac{1}{2}\left(\frac{\alpha(8) - \alpha(4)}{4} + \frac{\alpha(4) - \alpha(0)}{-4}\right)$ etc.

We expect $\alpha'(76) = \alpha'(0)$ since the motion is periodic, so this only needs to be done with $k = 0, 1, 2, \ldots, 18$

7. If the object moves at constant speed, then
(4) $\|\alpha'(t)\| = k$ for some constant k. Hence $\|\alpha'(t)\|^2 = k^2$,
so $\alpha'(t) \cdot \alpha'(t) = k^2$. Use the formula from 3(c)
from this section, we get
(4)

$$0 = \alpha'(t) \cdot \alpha''(t) + \alpha''(t) \cdot \alpha'(t) = 2\,\alpha'(t) \cdot \alpha''(t)$$

this says $v(t) = \alpha'(t)$ and $a(t) = \alpha''(t)$ are <u>orthogonal</u>.

Examination of the symmetry of the table also shows that

$(x(4k), y(4k)) = (x(76-4k), -y(76-4k))$, so

we can derive our approximations for the points in the second half of the list from those for the points in the first half by changing some signs (☺)

(b) Proceeding as in (a) we get

$\alpha'(0) \doteq (3.09, 1.0925)$

$\alpha'(4) \doteq (1.69, .0525)$

$\alpha'(8) \doteq (1.2325, -.0625)$

$\alpha'(12) \doteq (.945, -.1175)$

$\alpha'(16) \doteq (.73, -.1450)$

$\alpha'(20) \doteq (.555, -.1675)$

$\alpha'(24) \doteq (.4, -.1775)$

$\alpha'(28) \doteq (.2625, -.1875)$

$\alpha'(32) \doteq (.1275, -.1925)$

$\alpha'(36) \doteq (0, -.19)$, then

$\alpha'(40) \doteq (-.1275, -.1925)$

$\alpha'(44) \doteq (-.2625, -.1875)$

$\alpha'(48) \doteq (-.4, -.1775)$

$\alpha'(52) \doteq (-.555, -.1675)$

$\alpha'(56) \doteq (-.73, -.1450)$

$\alpha'(60) \doteq (-.945, -.1175)$

$\alpha'(64) \doteq (-1.2325, -.0625)$

$\alpha'(68) \doteq (-1.69, .0525)$

$\alpha'(72) \doteq (-3.09, 1.0925)$

and $\alpha'(76) \doteq \alpha'(0)$.

10 points

($\frac{1}{2}$ each)

don't take off for small arithmetic errors, etc.

(c) the comet is moving fastest at perihelion $(t=0, 76)$,
(2) and most slowly at aphelion $(t \doteq 38)$

(d) this differs from an ellipse as in Example 2.4B
$\alpha(t) = (a\cos t, b\sin t)$ because there the speed is the same at both end points of the major axis.
(2) Say $a > b$. Then $\alpha(0) = (a, 0)$ and $\alpha'(0) = (0, b)$ while $\alpha(\pi) = (-a, 0)$ and $\alpha'(\pi) = (0, -\pi)$. Hence $\|\alpha'(0)\| = \|\alpha'(\pi)\|$. the position of the ellipse is also different — sun $(0,0)$ is at one focus.

Comment: Kepler's Laws apply in this situation too, and the second law says the vector from the sun to the comet sweeps out equal areas in equal times. this implies what we said above: the speed is fastest when the comet is at its closest approach to the sun, and slowest when it is farthest away.