MATH 241 PS 2 Solutions

§ 1.3 /

1. (b) $u = (1, \sqrt{3}) \quad v = (\sqrt{3}, 1)$

$$\boxed{u \cdot v = 2\sqrt{3}}$$

$$\cos\theta = \frac{2\sqrt{3}}{\|u\|\,\|v\|} = \frac{2\sqrt{3}}{\sqrt{1+3}\cdot\sqrt{3+1}} = \frac{\sqrt{3}}{2}$$

So $\boxed{\theta = \frac{\pi}{6}}$

(d) $u = (\sqrt{2}, \sqrt{2}, 1) \qquad v = (1, 0, \sqrt{2})$

$$\boxed{u \cdot v = 2\sqrt{2}}$$

$$\cos\theta = \frac{2\sqrt{2}}{\sqrt{2+2+1}\cdot\sqrt{1+2}} = \frac{2\sqrt{2}}{\sqrt{15}} \doteq .73$$

$$\theta = \cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{15}}\right) \doteq .75 \text{ rad}$$

3. (a) Let $u = (a, b)$ $v = (c, d)$. Then

$$u \cdot v = ac + bd = ca + db = v \cdot u,$$

where the middle equation follows by commutativity of multiplication in $\mathbb{R}$.

(b) (i) Now let $w = (e, f)$. Then

$$\begin{aligned} u \cdot (v + w) &= (a, b) \cdot (c+e, d+f) \\ &= a(c+e) + b(d+f) \\ &= ac + ae + bd + bf \quad \text{(distributive law)} \\ &= (ac + bd) + (ae + bf) = u \cdot v + u \cdot w. \end{aligned}$$

(ii) If $c \in \mathbb{R}$ is a scalar, $u = (a, b)$, $v = (d, e)$

$$(cu) \cdot v = (ca, cb) \cdot (d, e)$$
$$= cad + cbe$$
$$= c(ad + be) \quad \text{(distributive law)}$$
$$= c(u \cdot v)$$

We can also rearrange differently like this

$$(cu) \cdot v = cad + cbe$$
$$= a(cd) + b(ce) \quad \text{(Commutativity + associativity of mult distributivity)}$$
$$= (a, b) \cdot (cd, ce)$$
$$= u \cdot (cv)$$

4. From Prop 1.4, $u \cdot v = \|u\| \|v\| \cos\theta$, where θ is the angle between u, v. then
$|u \cdot v| = \|u\| \|v\| |\cos\theta| \le \|u\| \|v\|$.
since $-1 \le \cos\theta \le 1$ for all θ.

5. We have

$$\|u + v\|^2 = (u + v) \cdot (u + v)$$
$$= u \cdot u + 2 u \cdot v + v \cdot v \quad \text{(by 3 above)}$$
$$= \|u\|^2 + 2 u \cdot v + \|v\|^2$$

(Note: $u \cdot v \le |u \cdot v| \le \|u\| \|v\|$)

$$\le \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 \quad \text{(by 4)}$$
$$= (\|u\| + \|v\|)^2$$

$\therefore \|u + v\| \le \|u\| + \|v\|$ by taking square roots.

7. (a) $(1,2,0) \times (-1,2,1) = (2,-1,4)$

(c) $(1,0,\frac{1}{2}) \times (-2,1,2) = (-\frac{1}{2},-3,1)$

8. (a) Let $\left.\begin{array}{l} u = (a,b,c) \\ v = (d,e,f) \end{array}\right\}$ in $\mathbb{R}^3$

then $u \times v = (bf-ce, -(af-cd), ae-bd)$

and $v \times u = (ec-fb, -(fa-dc), db-ea)$

By examining the components, $ec-fb = -(bf-ce)$ and so forth. So $v \times u = -(u \times v)$ and $u \times v = -(v \times u)$.

(b) $u \times u = (bc-bc, -(ac-ac), ab-ab)$
$= (0,0,0)$.

9.

(a) We have

$\|(1,1,0)-(0,0,0)\| = \sqrt{2}$

$\|(1,0,1)-(0,0,0)\| = \sqrt{2}$

$\|(0,1,1)-(0,0,0)\| = \sqrt{2}$

$\|(1,1,0)-(1,0,1)\| = \|(0,1,-1)\| = \sqrt{2}$

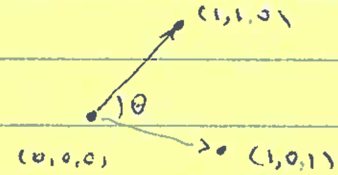
$\|(1,1,0)-(0,1,1)\| = \|(1,0,-1)\| = \sqrt{2}$

$\|(1,0,1)-(0,1,1)\| = \|(1,-1,0)\| = \sqrt{2}$

Hence all edges of the tetrahedron with these vertices have the same length

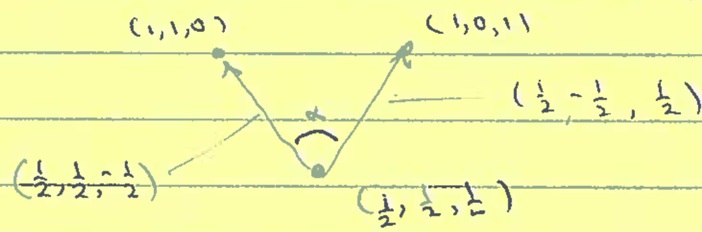
(b) the angles are all equal since they are angles in equilateral triangles: $\boxed{\theta = \frac{\pi}{3}}$

Check this as follows:



$$\cos\theta = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \quad \text{so} \quad \theta = \frac{\pi}{3} \text{ again.}$$

(c) We want the angle α, say, here:



$$\cos\alpha = \frac{\left(\frac12, \frac12, -\frac12\right)\cdot\left(\frac12, -\frac12, \frac12\right)}{\left\|\left(\frac12, \frac12, -\frac12\right)\right\| \, \left\|\left(\frac12, -\frac12, \frac12\right)\right\|}$$

$$= \frac{-1/4}{\sqrt{\frac34}\cdot\sqrt{\frac34}}$$

$$= \frac{-1}{3}$$

$$\alpha = \cos^{-1}\left(-1/3\right)$$

$$\doteq 1.91 \text{ rad}$$

$$= 109.5^\circ$$

§1.4

1 (c) $\{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = (3,2,0) + t(3,1,2), \ t \in \mathbb{R}\}$

same as given line.

2. (b) $\begin{cases} 1+3t = -2 \\ 3-t = 4 \end{cases}$

when $\boxed{t = -1}$, so <u>yes</u> Q is on L.

(d) $\begin{cases} 1+t = 1 \\ 3-2t = 2 \\ -1-t = 1 \end{cases}$

has <u>no</u> solution (first says $t = 0$, but that does not solve the others)

so <u>No</u>, Q is not on L.

4. the line through (x_0, y_0) with direction vector (v_1, v_2) has parametric equations

$$x = x_0 + t v_1$$
$$y = y_0 + t v_2$$

Since $v_1 \neq 0$, can solve the first to get

$t = \dfrac{x - x_0}{v_1}$ Substitute in second to get

$$y - y_0 = \frac{v_2}{v_1}(x - x_0)$$

this is the point-slope form with $m = \dfrac{v_2}{v_1}$.

If $v_1 = 0$ then $x = x_0$: gives a vertical line

7. (a) Given the line L_1 with

equations
$$\begin{cases} x = a + tb \\ y = c + td \\ z = e + tf \end{cases}$$

and L_2 a similar line with equations

$$\begin{cases} x = a' + ub' \\ y = c' + ud' \\ z = e' + uf' \end{cases}$$

(note $u \neq t$ is possible at an intersection!)

then we need to determine if the simultaneous equations

$$\begin{cases} a + tb = a' + ub' \\ c + td = c' + ud' \\ e + tf = e' + uf' \end{cases}$$

have a common solution.

b. (i) Equations are

$$-1 = 3 + 2u \Rightarrow u = -2 \text{ so}$$
$$3 + (-1)t = 4 + u \qquad \begin{cases} 3 - t = 2 \\ 2 + t = 3 \end{cases}$$
$$2 + t = -1 - 2u$$

$t = 1$, $u = -2$ is a solution so yes, the lines meet at $(-1, 2, 3)$

(ii) Equations are

$$\begin{cases} 1 - t = 2u \\ 2t = 2 + u \\ -2 + t = 1 - u \end{cases}$$

First two are $\begin{cases} t + 2u = 1 \\ 2t - u = 2 \end{cases}$

which has one solution $t = 1$, $u = 0$

But that does not solve the third equation.

So no, the lines do not meet.

8. (a) Proceed as in 7 to determine whether the lines meet, then use the formula $\cos\theta = \dfrac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}$ to determine the angle.

(b) (iv) the equations are

$$\left.\begin{aligned} -1 + 3t &= 3u \\ 2 - t &= 1 - 3u \end{aligned}\right] \Rightarrow \quad 1 - 3t + 1 = 2 - t$$

so $\boxed{t = 0, \quad u = -\frac{1}{3}}$

$$1 + 2t = 3 + 6u$$

$\therefore$ the lines do intersect. angle is

$$\cos\theta = \frac{(3,-1,2)\cdot(1,-3,6)}{\|(3,-1,2)\| \; \|(1,-3,6)\|}$$

$$= \frac{18}{\sqrt{14} \cdot \sqrt{46}} = \frac{9}{\sqrt{7} \cdot \sqrt{23}}$$

$$\theta = \cos^{-1}\left(\frac{9}{\sqrt{7} \cdot \sqrt{23}}\right)$$

11. (a) $1 \cdot (x - 1) + 0 \cdot (y + 1) + 1 \cdot (z - 2) = 0$

or $\boxed{x + z = 3}$

(d)

$$v_1 = (-2, 3, 2) - (1, 0, 2)$$
$$= (-3, 3, 0)$$
$$v_2 = (1, -1, 0) - (1, 0, 2)$$
$$= (0, -1, -2)$$

$$n = v_1 \times v_2 = (-6, -(+6), 3)$$
$$= (-6, -6, 3)$$
$$= 3(-2, -2, 1)$$

(Can use either n or $(-2, -2, 1)$!)

$$-2(x-1) + (-2)\cdot(y-0) + 1\cdot(z-2) = 0$$

or $\boxed{2x + 2y - z = 0}$

13 (a) $2\cdot 1 - 2 + 3(-1) = -3 \neq 0$, so Q is not in the plane. Using formula derived in class, the distance from $Ax + By + Cz = D$ to (x_0, y_0, z_0) is $\dfrac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$

$N = (2, -1, 3)$
$D = 4$
$Q = (1, 2, -1)$

$$= \frac{|N \cdot Q - D|}{\|N\|} \quad \text{for } N = (A, B, C)$$

Distance is $\dfrac{|-3-4|}{\sqrt{4+1+9}} = \dfrac{7}{\sqrt{14}} = \boxed{\dfrac{\sqrt{7}}{\sqrt{2}}}$