①

MATH 241 — PS 10 Solutions   ~~total~~ (104)

8, 6, 12, 6, 6, 12, 8, 8, 12, 12, 10, 4

§5.5

full credit just for this

2. (a) $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le k,\ z^2 + r^2 = k^2\}$

(2)

(b) $\{(r,\theta,z) \mid \tan\theta = k,\ r \geq 0\}$

(2)

or $\theta = \tan^{-1}(k)$

(c) $\{(r,\theta,z) \mid z = r^2 + k,\ 0 \le \theta \le 2\pi\}$

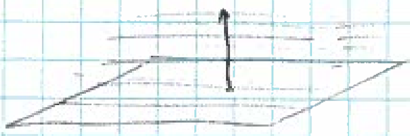
(2)

(d) $\{(r,\theta,z) \mid z = kr,\ 0 \le \theta \le 2\pi\}$

(2)

3. (a)

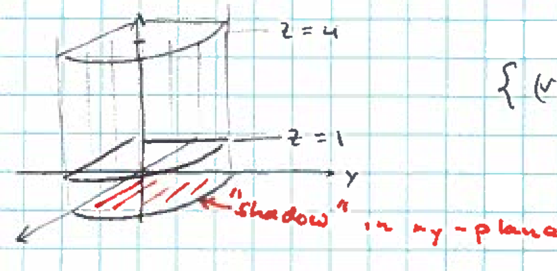
(2)



$\{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ r \geq 0,\ z \geq 0\}$

(c)

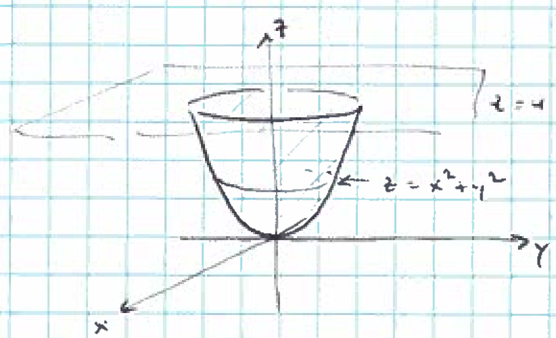
(2)



$\{(r,\theta,z) \mid 0 \le \theta \le \frac{\pi}{2},\ r \geq 0,\ 1 \le z \le 4\}$

(e)

(2)



$\{(r,\theta,z) \mid 4 \geq z \geq r^2,\ 0 \le \theta \le 2\pi\}$

7 (a)

$$\int_0^{\pi}\int_0^{2}\int_0^{3} r\cos\theta \cdot r\sin\theta \cdot z\, r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{2}\int_0^{3} r^3 \cos\theta \sin\theta \cdot z\, dz\, dr\, d\theta$$

$$= \left.\frac{z^2}{2}\right|_0^3 \cdot \left.\frac{1}{2}\sin^2\theta\right|_0^{\pi} \cdot \left.\frac{r^4}{4}\right|_0^2 = \frac{9}{2}\cdot 0 \cdot 4$$

(6)

$$= \boxed{0}$$

(C) $$\int_0^{2\pi}\int_2^4\int_1^3 \sin(r^2)\, r \, dz\, dr\, d\theta$$

$$= 2\pi \cdot 2 \cdot \tfrac{1}{2}\int_2^4 \sin(r^2)\, 2r\, dr$$

(6) $$= 4\pi \cdot \tfrac{-1}{2}\cos(r^2)\Big|_2^4$$

$$= -2\pi\left(\cos(16) - \cos(4)\right) \doteq 1.91$$

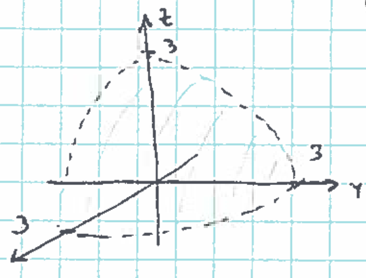
$$= \boxed{2\pi\left[\cos(4) - \cos(16)\right]}$$

§ 5.6

On these, verbal description w/o sketch is OK too

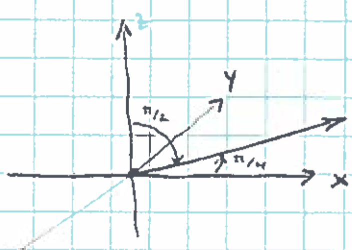
2. (a) the portion of the sphere of radius 3 ($\rho = 3$) in the first octant, but not including the boundary curves in the xy-, xz-, yz-planes.

(2)



(c) the ray $y = x$, $z = 0$, $x, y \geq 0$ in the xy-plane

(2)

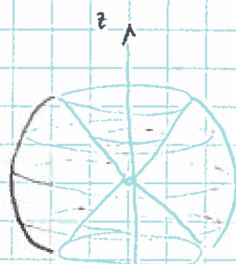


cute axes!

(e) the portion of the solid ball of radius 1, center $(0,0,0)$ between the cones $z = \sqrt{x^2+y^2}$ and $z = -\sqrt{x^2+y^2}$

$\varphi = \pi/4$ $\qquad$ $\varphi = \frac{3\pi}{4}$

(2)



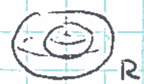
3 (a) $\{(\rho, \varphi, \theta) \mid \rho = k, \quad 0 \le \varphi \le \pi/2, \quad 0 \le \theta \le 2\pi\}$ 2

(b) $\{(\rho, \varphi, \theta) \mid \cot(\varphi) = k\}$ 2

or $\tan\varphi = \frac{1}{k}$

(c) $\{(\rho, \varphi, \theta) \mid 0 \le \rho \le 2, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \varphi \le \pi/2\}$ 2

5 (a)



$$\text{total heat} = \delta C \int_0^{2\pi} \int_0^{\pi} \int_1^2 \ln(\rho^2)\, \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$= \delta C \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\varphi \, d\varphi \cdot \int_1^2 \underbrace{2\rho^2}_{dv} \underbrace{\ln(\rho)}_{u} \, d\rho \quad \text{parts}$$

$$= \delta C \cdot 2\pi \cdot -\cos\varphi \Big|_0^{\pi} \cdot \left[\frac{2}{3}\rho^3 \ln(\rho) \Big|_1^2 - \int_1^2 \frac{2}{3}\rho^3 \cdot \frac{1}{\rho}\, d\rho \right]$$

6

$$= 4\pi\, \delta C \cdot \left[\frac{16}{3}\ln(2) - \frac{2}{9}\rho^3 \Big|_1^2 \right]$$

$$= 4\pi\, \delta C \cdot \left[\frac{16}{3}\ln(2) - \frac{16}{9} + \frac{2}{9} \right]$$

$$= \boxed{4\pi\, \delta\, C \left[\frac{16}{3}\ln(2) - \frac{14}{9} \right]}$$

(b) "right hemisphere" means hemisphere with $y \geq 0$

$$\text{total heat} = \delta C \cdot \int_0^{\pi} \int_0^{\pi} \int_1^3 \underbrace{\rho \sin\varphi \cos\theta}_{x} \cdot \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$= \delta C \cdot \underbrace{\int_0^{\pi} \cos\theta \, d\theta}_{=0} \cdot \int_0^{\pi} \sin^2\varphi \, d\varphi \cdot \int_1^3 \rho^3 \, d\rho$$

6

$$= \boxed{0}$$

§ 6.1 2 (b) $s = \int_0^{\pi} \sqrt{(-3\sin(3t))^2 + (3\cos(3t))^2 + 1} \; dt$

4

$$= \int_0^{\pi} \sqrt{9\left(\sin^2(3t) + \cos^2(3t)\right) + 1} \; dt$$

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$$= \int_0^{\pi} \sqrt{10}\ dt$$

$$= \boxed{\pi\sqrt{10}}.$$

(c) $$s = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2 + (3t^3)^2}\ dt$$

$$= \int_0^2 3t\sqrt{4 + 4t^2 + t^4}\ dt$$

Note: $t^4 + 4t^2 + 4 = (t^2+2)^2$

$$= \int_0^2 3t\,(t^2+2)\ dt$$

4

$$= \int_0^2 3t^3 + 6t\ dt$$

$$= \frac{3t^4}{4} + 3t^2 \Big|_0^2$$

$$= 12 + 12 - 0 = \boxed{24}$$ ✓

3. (a) line segment $(1,-1,2)$ to $(3,0,1)$ is $\alpha(t) = (1+2t, -1+t, 2-t)$ $t = 0..1$

$$\int_C f\,ds = \int_0^1 e^{(1+2t)+(-1+t)+(2-t)}\ \|(2,1,-1)\|\ dt$$

$$= \int_0^1 e^{2+2t}\sqrt{6}\ dt$$

4

$$= \frac{\sqrt{6}}{2} e^{2t+2} \Big|_0^1$$

$$= \boxed{\frac{\sqrt{6}}{2}\left(e^4 - e^2\right)}$$

(c) $\alpha(t) = (2\cos t, 2\sin t)$ $0 \le t \le 2\pi$

$\|\alpha'(t)\| = \|(-2\sin t, 2\cos t)\| = 2$ all t

4 $$\int_C f\,ds = \int_0^{2\pi} (2\cos t)^2 (2\sin t)\cdot 2\ dt$$

$$= 16\int_0^{2\pi} \cos^2 t \sin t\ dt = -\frac{16}{3}\cos^3 t\Big|_0^{2\pi} = \boxed{0}$$

§6.2

1\. (a) $\int_\alpha F\cdot T\,ds = \int_0^1 (-e^{-t}, e^t)\cdot(e^t, -e^{-t})\,dt$

$= \int_0^1 -1-1\,dt$

4

$= -2t\Big|_0^1 = \boxed{-2}$

(b) $\int_\alpha F\cdot T\,ds = \int_0^{2\pi} (\cos^3 t, \sin^3 t)\cdot(-3\cos^2 t\sin t, 3\sin^2 t\cos t)\,dt$

$= \int_0^{2\pi} -3\cos^5 t\sin t + 3\sin^5 t\cos t\,dt$

4

$= \frac{1}{2}\cos^6 t + \frac{1}{2}\sin^6 t\Big|_0^{2\pi}$

$= \boxed{0}$

(c) $\int_\alpha F\cdot T\,ds = \int_0^3 (2t, t, t)\cdot(1, 1, 2t)\,dt$

$= \int_0^3 3t + 2t^2\,dt$

$= \frac{3}{2}t^2 + \frac{2}{3}t^3\Big|_0^3$

4

$= \frac{27}{2} + 18 - 0$

$= \boxed{\frac{63}{2}}$

2\. (a) $\alpha(t) = (2\sin t, 2\cos t)$, $0\le t\le 2\pi$ clockwise from $(0,2)$

$\int_\alpha F\cdot T\,ds = \int_0^{\pi}((2\sin t, 2\cos t))\cdot(2\cos t, -2\sin t)\,dt$

$= \int_0^{2\pi} 0\,dt$

4

$= \boxed{0}$

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(b) $\alpha(t) = (t, t^2) \quad 0 \le t \le 2$

$$\int_\alpha F \cdot T \, ds = \int_0^2 (t^4, t^2) \cdot (1, 2t) \, dt$$

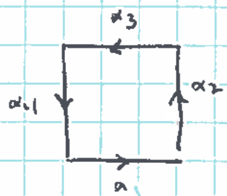
$$= \int_0^2 t^4 + 2t^3 \, dt$$

4

$$= \frac{t^5}{5} + \frac{1}{2} t^4 \Big|_0^2$$

$$= \frac{32}{5} + 8 - 0 = \boxed{\frac{72}{5}}$$

(C)



$\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$

$\alpha_1(t) = (t, 0)$, $\alpha_2(t) = (1, t)$, $\alpha_3(t) = (1-t, 1)$, $\alpha_4(t) = (0, 1-t)$

all $0 \le t \le 1$

So $$\int_\alpha F \cdot T \, ds = \int_0^1 (t^2, 0) \cdot (1, 0) \, dt + \int_0^1 (1, t) \cdot (0, 1) \, dt + \int_0^1 ((1-t)^2, (1-t)) \cdot (-1,$$

$$+ \int_0^1 (0, 0) \cdot (0, -1) \, dt$$

4

$$= \int_0^1 t^2 \, dt + \int_0^1 t \, dt + \int_0^1 -(1-t)^2 \, dt + 0$$

$$\frac{(1-t)^3}{3} \Big|_0^1$$

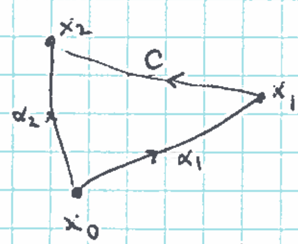
$$= \frac{1}{3} + \frac{1}{2} + \left(-\frac{1}{3}\right)$$

$$= \boxed{\frac{1}{2}}$$

4. $f(x) = \int_\alpha F \cdot T \, ds$ where α is any curve from x_0 to x

4

(a)



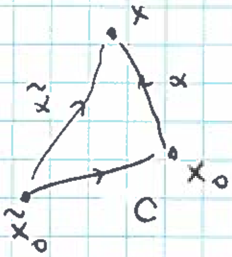
By path independence,

$$\int_{\alpha_1 \cup C \cup (-\alpha_2)} F \cdot T \, ds = 0$$

So $$\int_{\alpha_1} F \cdot T \, ds + \int_C F \cdot T \, ds - \int_{\alpha_2} F \cdot T \, ds = 0$$

$$\therefore \int_C F \cdot T \, ds = \int_{\alpha_2} F \cdot T \, ds - \int_{\alpha_1} F \cdot T \, ds = f(x_2) - f(x_1)$$

(b) + (c)



If $f(x) = \int_{\alpha} F \cdot T \, ds$ and $\tilde{f}(x) = \int_{\tilde{\alpha}} F \cdot T \, ds$,

then by independence of path,

$$\tilde{f}(x) = \int_{\tilde{\alpha}} F \cdot T \, ds$$

6

$$= \int_{C} F \cdot T \, ds + \underbrace{\int_{\alpha} F \cdot T \, ds}_{= f(x)}$$

So $\tilde{f}(x) - f(x) = \int_{C} F \cdot T \, ds$. this is constant since F has the independence of path property. Any C that is used yields the same, constant value.

5. (a) As shown in theorem 6.2, if $f(x) = \int_{\alpha} F \cdot T \, ds$ where α is any curve from x_0 to x, then $F = \nabla f$. So if $F = (F_1, F_2)$, we have $F_1 = \frac{\partial f}{\partial x}$ and $F_2 = \frac{\partial f}{\partial y}$

4 But then $\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial F_2}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$. The mixed partials of f are equal if we assume that F is differentiable with continuous partial derivatives of F_1, F_2. Hence $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. //