MATH 241 – Multivariable Calculus
Sample Exam Questions for Exam 1 – Solutions
September 23, 2013

I. Consider the set $Q = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 - y^2/4 - z^2 = 1\}$.

A) Identify the slices of Q in planes parallel to the three coordinate planes. Take $x, y, z = -2, -1, 0, 1, 2$ (at least).
Solution: The slices by the planes $x = c$ have equations $y^2/4 + z^2 = c^2 - 1$. These are ellipses if $c > 1$ or $c < -1$, single points if $c = \pm 1$ and empty if $-1 < c < 1$. The slices by planes $y = c$ have equations $x^2 - z^2 = 1 + c^2/4$. These are hyperbolas for all c in the planes parallel to the xz-plane). The slices by planes $z = c$ have equations $x^2 - y^2/4 = 1 + c^2$. These are also hyperbolas.

B) Use that information to generate a rough sketch of Q or a verbal description.
Solution: See the Maple plot posted with these solutions. A verbal description would go something like this: The surface consists of two sheets: one with $x \geq 1$ and the other with $x \leq -1$. There are no points on the surface with $-1 < x < 1$. The two sheets both have elliptical cross-sections by planes $x = c$. This is called a hyperboloid of two sheets because the sections by planes $y = c$ and $z = c$ are both hyperbolas.

C) Show that the image of the parametric curve $\alpha(t) = ((e^t + e^{-t})/2, 0, (e^t - e^{-t})/2)$ for t in $\mathbf{R}$ lies entirely in Q.
Solution: Substituting $x = (e^t + e^{-t})/2$, $y = 0$, and $z = (e^t - e^{-t})/2$ into the equation of Q we have

$$\begin{aligned} x^2 - y^2/4 - z^2 &= \big((e^t + e^{-t})/2\big)^2 - 0 - \big((e^t - e^{-t})/2\big)^2 \\ &= (e^{2t} + 2 + e^{-2t})/4 - (e^{2t} - 2 + e^{-2t})/4 \\ &= 1. \end{aligned}$$

Therefore all points on the image of α satisfy the equation of Q, so the curve lies entirely in the surface.

II.

A) Find the equation of the plane in $\mathbf{R}^3$ containing the point $P = (1, 1, 1)$ and the line through $Q = (0, 1, -1)$ with direction vector $v = (-4, 1, 2)$.
Solution: The plane contains the vector $(-4, 1, 2)$ and $P - Q = (1, 0, 2)$. Hence a normal vector can be found by computing the cross product

$$N = (-4, 1, 2) \times (1, 0, 2) = (2, 10, -1)$$

The plane has equation

$$N \cdot (x - 1, y - 1, z - 1) = 2x + 10y - z - 11 = 0.$$

B) The points $P, Q, Q+v$ from part A are three corners of a parallelogram in the plane you found. Find the fourth corner and the area of that parallelogram.
Solution: The fourth corner is $P+v=(-3,2,3)$. The area of the parallelogram is

$$\|(-4,1,2)\times(1,0,2)\| = \|(2,10,-1)\| = \sqrt{4+100+1} = \sqrt{105}.$$

C) Two lines with direction vectors $v=(0,1,2)$ and $w=(4,3,2)$ meet at $(0,0,0)$. Find the acute angle between the two lines.
Solution: The angle θ satisfies:

$$\cos\theta = \frac{v\cdot w}{\|v\|\|w\|} = \frac{7}{\sqrt{5}\sqrt{29}}.$$

Hence $\theta = \cos^{-1}\left(\frac{7}{\sqrt{5}\sqrt{29}}\right) \doteq 1.31$ radians.

D) Show that if $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)$, and $w=(w_1,w_2,w_3)$ are general vectors in space, then $u\times(v+w) = u\times v + u\times w$.
Solution: We have, after applying the distributive law and splitting up the vector into a sum of two other vectors:

$$\begin{aligned}
u\times(v+w) &= (u_1,u_2,u_3)\times(v_1+w_1, v_2+w_2, v_3+w_3)\\
&= (u_2(v_3+w_3)-u_3(v_2+w_2), -(u_1(v_3+w_3)-u_3(v_1+w_1)),\\
&\qquad u_1(v_2+w_2)-u_2(v_1+w_1))\\
&= (u_2v_3-u_3v_2+u_2w_3-u_3w_2, -(u_1v_3-u_3v_1)-(u_1w_3-u_3w_1),\\
&\qquad u_1v_2-u_2v_1+u_1w_2-u_2w_1)\\
&= (u_2v_3-u_3v_2, -(u_1v_3-u_3v_1), u_1v_2-u_2v_1)+\\
&\qquad (u_2w_3-u_3w_2, -(u_1w_3-u_3w_1), u_1w_2-u_2w_1)\\
&= u\times v + u\times w.
\end{aligned}$$

III.

A) Sketch the parametric curve $\alpha(t) = (4\cos(\pi t), 3\sin(\pi t))$ for $0\le t\le 1$. Show the starting and final points and indicate the direction the curve is traced with an arrow.
Solution: This is the top half of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, starting at $\alpha(0)=(4,0)$ and ending at $\alpha(1)=(-4,0)$. The curve is traced in the counter-clockwise direction.

B) Give a parametrization of the circle with center $(1,3)$ and radius $r=4$ so that one full circuit of the circle is traced out clockwise for $0\le t\le 4\pi$.
Solution: $\beta(t) = (4\cos(t/2)+1, -4\sin(t/2)+3)$ is one such parametrization. starting from $\beta(0)=(5,3)$.) Another curve satisfying the given conditions is $\gamma(t) = (4\sin(t/2)+1, 4\cos(t/2)+3)$. This one starts from $\gamma(0)=(1,7)$.

IV. All parts of this problem refer to the parametric curve

$$\alpha(t) = ((1+\cos(t))\sin(t), -(1+\cos(t))\cos(t)),$$

called a cardioid (see the Maple plot).

A) Show that for all t, the distance from the origin to the point $\alpha(t)$ is given by $1+\cos(t)$.
Solution: The distance is the same as the magnitude $\|\alpha(t)\|$. Computing this, by means of the basic trig identity, we find:

$$\begin{aligned}
\|\alpha(t)\| &= \sqrt{(1+\cos(t))^2\sin^2(t) + (1+\cos(t))^2\cos^2(t)} \\
&= \sqrt{(1+\cos(t))^2(\sin^2(t)+\cos^2(t))} \\
&= \sqrt{(1+\cos(t))^2} \\
&= |1+\cos(t)| \\
&= 1+\cos(t),
\end{aligned}$$

since $\cos(t) \geq -1$ for all t, so $1+\cos(t) \geq 0$.

B) At how many different times t in $[0, 2\pi]$ is the y-coordinate of $\alpha(t)$ equal to zero? Find them.
Solution: The y-coordinate is $-(1+\cos(t))\cos(t)$. This is only $= 0$ if either $\cos(t) = -1$ or $\cos(t) = 0$. There are three such t values in the given interval: $t = \pi/2, \pi, 3\pi/2$.

C) Find a parametrization of the tangent line to the cardioid at the point $\alpha(\pi/4)$.
Solution: The point on the cardioid is

$$\alpha(\pi/4) = ((1+\sqrt{2}/2)\sqrt{2}/2, -(1+\sqrt{2}/2)\sqrt{2}/2) = \left(\frac{1+\sqrt{2}}{2}, -\frac{1+\sqrt{2}}{2}\right).$$

The tangent vector at a general $\alpha(t)$ is

$$\alpha'(t) = (-\sin^2(t) + \cos^2(t) + \cos(t), 2\sin(t)\cos(t) + \sin(t)).$$

So $\alpha'(\pi/4) = (\sqrt{2}/2, 1+\sqrt{2}/2)$. The tangent line is

$$\lambda(u) = \alpha(\pi/4) + u\alpha'(\pi/4) = \left(\frac{1+\sqrt{2}}{2}, -\frac{1+\sqrt{2}}{2}\right) + u\left(\frac{\sqrt{2}}{2}, 1+\frac{\sqrt{2}}{2}\right).$$

D) Give a parametrization for any one circle whose interior region completely contains the cardioid. Explain how you determined your center and radius.
Solution: By part A, every point on the cardioid is distance at most 2 from the origin. So if we take any radius $r > 2$ and center $(0, 0)$, the cardioid will be contained in the interior region of the circle: $\delta(t) = (3\cos(t), 3\sin(t))$ gives one such circle if we take $t \in [0, 2\pi]$.