PS9: 5.4 / 4a, 6b, 8a ; 5.5 / 7c ; 5.6 / 2ace, 5a ; 6.4 / 1a, 10, 11, 12

4 4 120 (also by cyl) 6 6 4 10 12 12

1

Problem Set 9 Solutions

84 total

5.4 / 4a.

4

$$\int_0^1\int_{-1}^1\int_0^2 x^2 - yz \, dz\, dy\, dx = \int_0^1\int_{-1}^1 x^2 z - \frac{yz^2}{2}\Big|_0^2 dy\, dx$$

$$= \int_0^1\int_{-1}^1 2x^2 - 2y \; dy\, dx$$

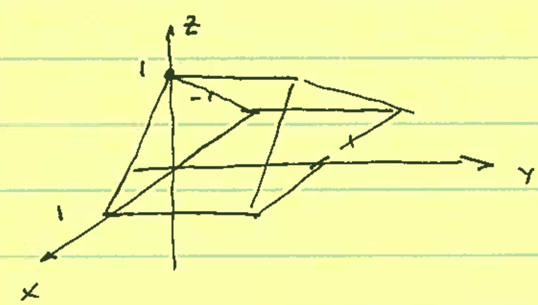
$$= \int_0^1 2x^2 y - y^2 \Big|_{-1}^1 dx$$

$$= \int_0^1 4x^2 \, dx$$

$$= \frac{4}{3}x^3\Big|_0^1 = \boxed{\frac{4}{3}}$$

2

6b. The prism looks like this

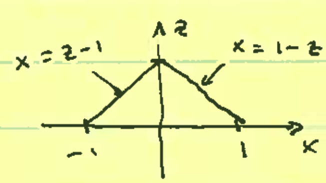


4

Note that the problem says: find a single iterated integral. The dz dy dx or dz dx dy integrals will not work for this since the "top" surface changes at x = 0. So we can project onto the xz plane to find the "shadow," and set it up like this:

$$\int_0^1\int_{z-1}^{1-z}\int_0^1 f(x,y,z)\; dy\, dx\, dz$$

1 2 1



"shadow" in xz-plane

$z = 1 - x^2 - y^2$ intersects the xy-plane $z=0$ when $x^2+y^2=1$, so the circular disk $\{(x,y) \mid x^2+y^2 \le 1\}$ is the "shadow"

8 a.

$$m = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} z \, dz\, dy\, dx \quad 2$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \tfrac{1}{2}(1-x^2-y^2)^2 \, dy\, dx$$

Can evaluate this using polar coordinates:

$$= \int_0^{2\pi} \int_0^1 \tfrac{1}{2}(1-r^2)^2 \, r\, dr\, d\theta$$

$$= \int_0^{2\pi} -\tfrac{1}{2}\cdot\tfrac{1}{6}(1-r^2)^3 \Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{12} \, d\theta = \boxed{\tfrac{\pi}{6}} \; 2 \; ✓$$

then

$$\bar{x} = \frac{6}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} x z \, dz\, dy\, dx \quad 2$$

$$= \frac{6}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x}{2}(1-x^2-y^2)^2 \, dy\, dx$$

$$= \frac{6}{\pi} \int_0^{2\pi} \int_0^1 \frac{r\cos\theta}{2} (1-r^2)^2 \, r\, dr\, d\theta$$

$$= \frac{6}{\pi} \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{= 0} \cdot \int_0^1 \tfrac{1}{2} r^2 (1-r^2)^2 \, dr$$

$$= \boxed{0} \quad 2$$

Similarly,

$$\bar{y} = \frac{6}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2} yz \, dz\,dy\,dx \quad 2$$

$$= \frac{6}{\pi}\underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{=0}\cdot\int_0^1 \tfrac{1}{2}r^2(1-r^2)^2\,dr$$

$$= \boxed{0} \quad 2 \checkmark$$

$$\bar{z} = \frac{6}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2} z^2\,dz\,dy\,dx \quad 2$$

$$= \frac{6}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \tfrac{1}{3}z^3\Big|_0^{1-x^2-y^2}\,dy\,dx$$

$$= \frac{6}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \tfrac{1}{3}(1-x^2-y^2)^3\,dy\,dx$$

polar again
$$= \frac{6}{\pi}\int_0^{2\pi}\int_0^1 \tfrac{1}{3}(1-r^2)^3\,r\,dr\,d\theta$$

$$= \frac{6}{\pi}\int_0^{2\pi} -\tfrac{1}{2}\cdot\tfrac{1}{4}\cdot\tfrac{1}{3}(1-r^2)^4\Big|_0^1\,d\theta$$

$$= \frac{6}{\pi}\int_0^{2\pi}\frac{1}{24}\,d\theta$$

$$= \frac{6}{\pi}\cdot\frac{\pi}{12} = \boxed{\frac{1}{2}} \quad 2 \checkmark$$

The cylindrical coordinate forms are almost the same:

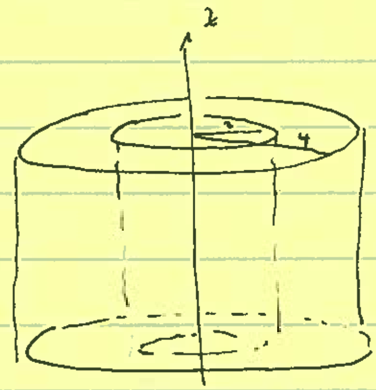
(4)

$$M = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} z\,r\,dz\,dr\,d\theta, \quad \underline{\text{etc}}.$$

$$\bar{x} = \frac{1}{M}\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\cos\theta\cdot z\cdot r\,dz\,dr\,d\theta, \ \ldots$$

5.5 / 7c

R :



$z = 3$

2

$z = 1$

$$\iiint_R \sin(x^2+y^2)\, dV = \int_0^{2\pi}\int_2^4\int_1^3 \sin(r^2)\, r \, dz\, dr\, d\theta$$

2

$$= \int_0^{2\pi}\int_2^4 2r\sin(r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} -\cos(r^2)\Big|_2^4 \, d\theta$$

$$= \int_0^{2\pi} -\cos(16) + \cos(4)\, d\theta$$

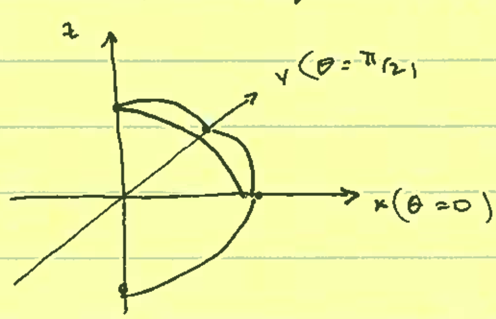
$$= \boxed{2\pi\left(\cos(4) - \cos(16)\right)}$$ ✓

2

$(\doteq 1.91)$

5.6 /

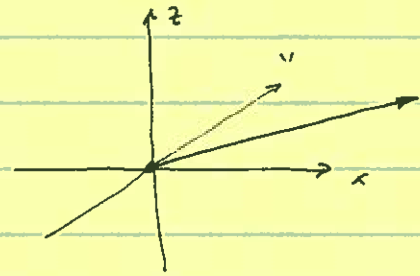
2 a) One quarter of the sphere of radius 3 centered at $(0,0,0)$, between xz and yz-plane



2

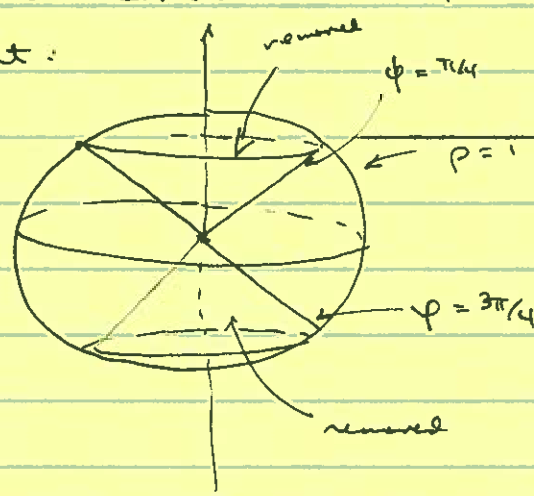
(think of the skin of a quarter of an orange)

c) A ray in xy-plane ($\phi = \pi/2$), along the line $y = x$ ($\theta = \pi/4$)

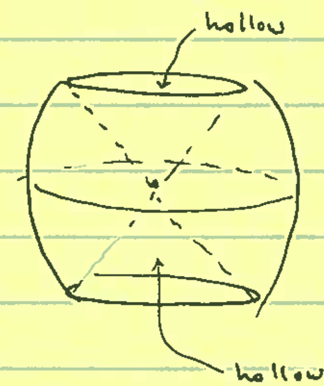


2

e) A solid sphere of radius 1, from which two conical "caps" have been drilled out:



leaving:



2

$u = \ln(\rho^2)$ $\quad du = \frac{2}{\rho}$

$dv = \rho^2 d\rho$ $\quad v = \frac{\rho^3}{3}$

5a. $\iiint_R \delta C T \, dV$

$$= \int_0^{2\pi} \int_0^{\pi} \int_1^{2} \delta C \overbrace{\ln(\rho^2)}^{u} \overbrace{\rho^2}^{dv} \sin\varphi \, d\rho \, d\varphi \, d\theta$$

(parts)

$$= \delta C \int_0^{2\pi} \int_0^{\pi} \sin\varphi \cdot \left[\frac{\rho^3 \ln(\rho^2)}{3} - \frac{2}{9}\rho^3 \right]_1^2 d\varphi \, d\theta$$

$$= \delta C \int_0^{2\pi} \int_0^{\pi} \sin\varphi \left(\frac{8}{3} \ln(4) - \frac{14}{9} \right) d\varphi \, d\theta$$

$$= \delta C \cdot \left(\frac{8}{3} \ln(4) - \frac{14}{9} \right) \cdot \int_0^{2\pi} -\cos\varphi \Big|_0^{\pi} \, d\theta$$

$$= \delta C \cdot 2 \cdot \left(\frac{8}{3} \ln(4) - \frac{14}{9} \right) \cdot \int_0^{2\pi} d\theta$$

$$= \boxed{\delta C \cdot 4\pi \cdot \left(\frac{8}{3} \ln(4) - \frac{14}{9} \right)}$$ 2 ✓

6.4 / 1a

2

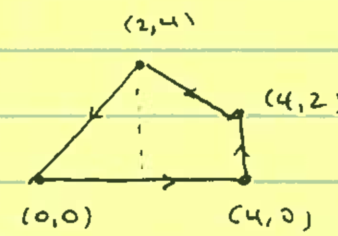
$$s = \int_0^1 \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} \, dt$$

$$= \int_0^1 \sqrt{e^{2t} (\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 t)}$$

$$= \int_0^1 \sqrt{2} \, e^t \, dt$$

$$= \boxed{\sqrt{2}(e - 1)}$$ 2

10 a) the quadrilateral is



Green's theorem says

$$\int_{\alpha} F \cdot T \, ds = \iint_R \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \, dA$$ 2

$$= \int_0^2 \int_0^{2x} y + 1 \, dy \, dx + \int_2^4 \int_0^{6-x} y + 1 \, dy \, dx$$ 2

$$= \int_0^2 \frac{4x^2}{2} + 2x \, dx + \int_2^4 \frac{(6-x)^2}{2} + 6 - x \, dx$$

$$= \int_0^2 2x^2 + 2x \, dx + \int_2^4 24 - 7x + \frac{x^2}{2} \, dx$$

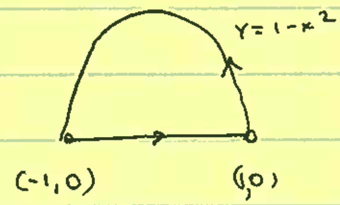
$$= \frac{2x^3}{3} + x^2 \Big|_0^2 + 24x - \frac{7x^2}{2} + \frac{x^3}{6} \Big|_2^4$$

$$= \frac{28}{3} + \frac{46}{3}$$

2

$$= \boxed{\frac{74}{3}}$$ ✓

b)



$$\int_\alpha F \cdot T \, ds = \iint_R \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \, dA$$

$$= \int_{-1}^{1} \int_0^{1-x^2} 1 + 1 \, dy \, dx \quad 2$$

$$= \int_{-1}^{1} 2(1-x^2) \, dx$$

$$= 2x - \frac{2x^3}{3} \Big|_{-1}^{1} = \boxed{\frac{8}{3}} \quad 2$$

11. (a)

using $A = \int_\alpha -y \, dx$ (i.e. $F(x,y) = (-y, 0)$)

$\alpha(t) = (\cos^3 t, \sin^3 t)$ 2

$t = 0 .. 2\pi$

$$A = \int_0^{2\pi} -\sin^3 t \cdot 3\cos^2 t \cdot (-\sin t) \, dt \quad 2$$

$$= 3\int_0^{2\pi} \sin^4 t \cos^2 t \, dt$$

$$= 3\left[\frac{\sin^5 t \cos t}{6} \Big|_0^{2\pi} + \frac{1}{6} \int_0^{2\pi} \sin^4 \, dt \right]$$

(8)

$$= 3\left[\frac{\sin^5 t\cos t}{6}\Big|_0^{2\pi} + \frac{1}{6}\int_0^{2\pi}\sin^4 t\,dt\right.$$

$$= \frac{1}{2}\int_0^{2\pi}\sin^4 t\,dt$$

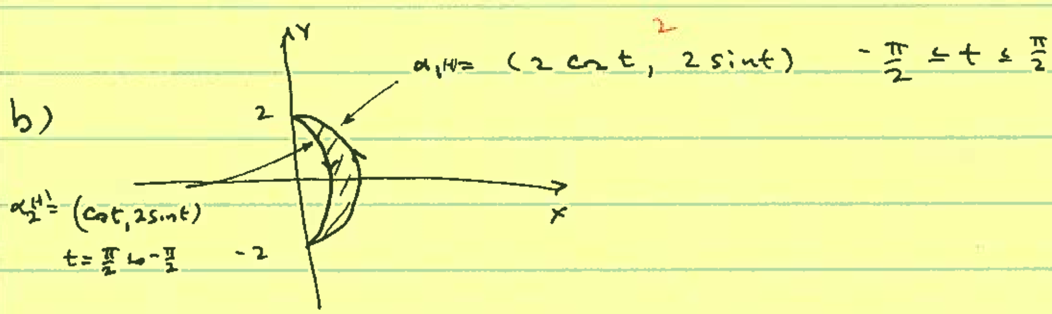
$$= \frac{1}{2}\left[-\frac{1}{4}\cancel{\sin^3 t\cos t}\Big|_0^{2\pi} + \frac{3}{4}\int_0^{2\pi}\sin^2 t\,dt\right]$$

$$= \frac{3}{8}\int_0^{2\pi}\sin^2 t\,dt$$

$$= \frac{3}{8}\left[\frac{t}{2} - \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= \boxed{\frac{3\pi}{8}}$$ ✓ 2

b)



$$A = -\int y\,dx$$

$$= -\int_{-\pi/2}^{\pi/2} 2\sin t\cdot(-2\sin t)\,dt$$ 2

$$+\int_{-\pi/2}^{\pi/2} 2\sin t\cdot\sin t\,dt$$

$$= 2\int_{-\pi/2}^{\pi/2}\sin^2 t\,dt$$

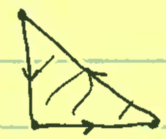
$$= 2\cdot\left[\frac{t}{2} - \frac{1}{4}\sin 2t\right]_{-\pi/2}^{\pi/2}$$

$$= \boxed{\pi}$$ ✓ 2

check $\frac{4\pi - 2\pi}{2}$ ✓

12.

a)



$$\text{Total flux} = \int_{\alpha} F \cdot N \, ds \qquad F = (y-1, x-1)$$

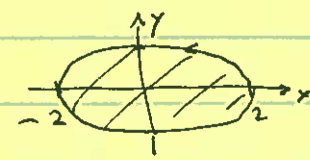
so $\frac{\partial}{\partial x}(y-1) + \frac{\partial}{\partial y}(x-1) = \underline{\underline{0}}.$

$$= \iint_R \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \, dA$$

$$= \int_0^3 \int_0^{3-x} 0 \, dA$$

$$= \boxed{0}$$

b) $\frac{x^2}{4} + y^2 = 1$

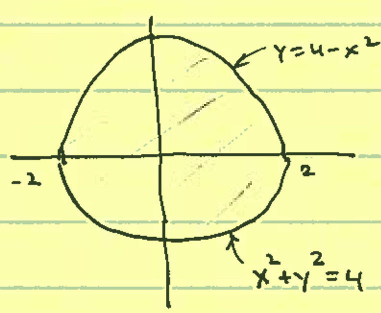


$$\text{Total flux} = \int_{\alpha} F \cdot N \, ds$$

$$= \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \underbrace{-1}_{= \frac{\partial u}{\partial x}} + \underbrace{1}_{= \frac{\partial v}{\partial y}} \, dA$$

$$= \boxed{0}$$

c)



$$\text{Total flux} = \int_{\alpha} F \cdot N \, ds$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{4-x^2} 2+2 \, dy \, dx$$

$$= 4 \cdot \text{area}(R) = 4\int_{-2}^{2} 4-x^2 + \sqrt{4-x^2} \, dx$$

$$= 4\left[4x - \frac{x^3}{3} + \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 4\left[16 - \frac{16}{3} + 0 + 2\pi \right]$$

$$= \boxed{\frac{128}{3} + 8\pi} \checkmark$$